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The role of the Pauli principle in three-cluster systems composed of identical clusters ⋆

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Abstract

Within the microscopic model based on the algebraic version of the resonating group method the role of the Pauli principle in the formation of continuum wave function of nuclear systems composed of three identical s-clusters has been investigated. Emphasis is placed upon the study of the exchange effects contained in the genuine three-cluster norm kernel. Three-fermion, three-boson, three-dineutron ($3d'$) and 3α systems are considered in detail. Simple analytical method of constructing the norm kernel for 3α system is suggested. The Pauli-allowed basis functions for the 3α and $3d'$ systems are given in an explicit form and asymptotic behavior of these functions is established. Complete classification of the eigenfunctions and the eigenvalues of the ^{12}C norm kernel by the ^{8}Be$=\alpha+\alpha$ eigenvalues has been given for the first time. Spectrum of the ^{12}C norm kernel is compared to that of the ^{5}H system.

Key words: three-cluster microscopic model, Pauli-allowed states, resonating-group method, neutron-rich nuclei
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1. Introduction

The question of the role of the Pauli principle in three-cluster systems goes back a long way in history. The ^{12}C nucleus has been established to exhibit three-alpha-cluster structure forty years ago. Since then many microscopic, macroscopic and semi-microscopic cluster models have been applied to analyze the structure of the ground and excited states of this nucleus. In particular, microscopic 3α calculation was performed within the resonating group method (RGM) by Kamimura [1] and within the generator coordinate method (GCM) by Uegaki [2]. Both calculations give reasonable results for the ground state of the ^{12}C and some excited states. However, wave functions provided by these models are very complicated and heavy to handle. Furthermore, although RGM ensures correct account of nucleon exchange between different clusters, an antisymmetry requirement on the total wave function can be violated by the improper truncation of model space. The measure of this violation is lacking. For example, in Ref. [1] the RGM calculation was performed with truncation to a space where the angular momentum l of $\alpha-\alpha$ relative wave function was fixed to be equal to zero. At the same time, even the lowest Pauli-allowed state of the ^{12}C represents the mixture of $l=0$, 2 and 4. This raises the question as to whether such truncation is consistent with the requirements of

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the Pauli principle. As for the GCM, seven-dimensional numerical integrals are involved in the calculation and the integration over generator coordinate is replaced into summation over some mesh-points. However, the generator coordinate is chosen to be real and its domain is not well-defined, while only complex generator parameters ensure the existence of inverse transition from the generator parameter space to the coordinate space [3].

A number of macroscopic models were also applied to studying the ^{12}C nucleus (see, for example, Refs. [4,5]). Within such models, clusters are considered to be structureless particles interacting via local potentials, which reproduce experimental $\alpha-\alpha$ phase shifts. But the antisymmetrization prevents $\alpha-\alpha$ potential from being local. Furthermore, an accurate elimination of the Pauli-forbidden states is needed to get any reliable results on the structure of low-lying states of ^{12}C nucleus. By this reason, all macroscopic models fail to reproduce main properties of the 3α system.

Orthogonality condition model (OCM) has been proposed in Ref. [6] as an approximation of resonating group method and applied to the investigation of the three-alpha structure in ^{12}C in Refs. [7,8]. In OCM the inter-cluster wave function is required to be orthogonal to the forbidden states and, consequently, the latter states are completely removed from the wave function. In two-cluster systems, OCM allows one to obtain only eigenfunctions of the antisymmetrization operator, not eigenvalues, which enter the Schrödinger equation and thus affect the dynamics of cluster system. Hence, in OCM a part of exchange effects is missing while it could be essential, especially in three-cluster systems. This is supported by the fact that the lack of binding energy of ^{12}C is observed in OCM compared to RGM calculations [9]. In Ref. [10] Horiuchi assumed that the 3α redundant solutions are all reducible to $\alpha-\alpha$ redundancy and the Pauli-allowed 3α wave functions can be defined from the requirement of their orthogonality to the $\alpha-\alpha$ forbidden states. Horiuchi also noticed that the Pauli-allowed basis functions should be classified by the number of total oscillator quanta and indices of the SU(3) symmetry (λ,μ), because the second-order Casimir operator of the SU(3) group commutes with the operator of permutation of the nucleon position vectors. However, most of the three-cluster states are SU(3)-degenerate and the degree of SU(3)-degeneration increases drastically with increasing the number of oscillator quanta. Pauli-allowed basis functions corresponding to the same SU(3)-symmetry indices differs in their eigenvalues and should be labeled by additional quantum number. But construction method of OCM does not specify the character of the latter quantum number and, hence, in the case of SU(3)-degeneration neither eigenvalues nor eigenfunctions of the antisymmetrization operator can be uniquely determined.

Recently Fujiwara and co-authors proposed a new type of three-cluster equation that employs a two-cluster RGM kernel for the inter-cluster interaction [11]. The authors applied this equation to systems composed of three dineutrons and three alpha-clusters. Results for binding energy for 3α system, obtained by diagonalization of the Hamiltonian using [3] symmetric translationary-invariant harmonic oscillator basis, have been compared to the RGM and OCM calculations. Energy of the ^{12}C ground state given in Ref. [11] is closer to experimental data than the value provided by OCM, but still underbound by 1.5 MeV compared to the 3α RGM. The reason is that the three-cluster equation used in Ref. [11] does not include some exchange effects contained in the genuine three-cluster norm kernel.

The three-cluster norm kernel contains complete information about the Pauli-allowed model space for the relative motion of clusters. Eigenvalues and eigenfunctions of the norm kernel are solely determined by the assumed internal wave functions of the clusters. Hence, careful analysis of these quantities is very useful for understanding the structure of three-cluster systems. However, there are few papers concerning this question. Eigenvalues and eigenfunctions of the norm kernels for 3α and ^{16}O+2α systems have been calculated in Ref. [12]. They have been characterized by the SU(3)-indices and an additional quantum number, but the meaning of the latter number has not been established and no classification with respect to this number has been provided. The eigenstates of the norm kernel were expanded by the harmonic oscillator wave functions in the rectangular coordinate representation. Only three simplest Pauli-allowed states of the ^{24}Mg have been given in Ref. [12], while eigenfunction of the norm kernel for ^{12}C have not been presented. Coincidence of some eigenvalues of the ^{24}Mg with the eigenvalues of the two-body norm kernels of ^{8}Be= $\alpha+\alpha$ and ^{20}Ne= α+^{16}O at large number of quanta has been noticed, but this property has not been taken advantage of in classification of the three-cluster Pauli-allowed states.

Summarizing, there are several problems still outstanding: construction of eigenfunctions of the antisym-

metrization operator for the three-cluster systems in simple, tractable form, complete classification of these states (i.e., resolving SU(3)-degeneration problem) and elaboration of the truncation method of the model space consistent with the requirements of the Pauli exclusion principle.

For the case of three-cluster systems composed of an s-cluster and two neutrons we have shown that all these problems can be resolved within the discrete representation of a complete basis of allowed states of the multiparticle harmonic oscillator (classified with the use of the SU(3) symmetry indices and defined in the Fock-Bargmann space) [13]. Classification of the eigenvalues of the three-cluster systems with the help of eigenvalues of the two-body subsystem was suggested in Ref. [13]. We observed that asymptotic behavior of basis functions consistent with the requirements of the Pauli principle gives an indication of possible decay channels of a three-cluster nucleus and allows us to specify the most important decay channels. Such asymptotic behavior corresponds rather to the subsequent decay of the three-cluster system than to the so-called "democratic decay" associated with the hyperspherical harmonics, which are widely used for the description of three-cluster systems.

In the present paper we extend our approach to the three-cluster systems composed of identical s-clusters. In Section 2 the theoretical model is briefly explained. Norm kernels of three-fermion and three-boson systems are considered in detail in Section 3 and Section 4, respectively. Section 5 is devoted to the analysis of the eigenvalues and eigenfunctions of the norm kernel of the three-dineutron system. The Pauli-allowed states of the 3α-system are constructed and classified in Section 6. The asymptotic behavior of these functions is established and compared to the asymptotic form of the eigenfunctions of the ${}^3\mathrm{H}+n+n$ norm kernel. Concluding remarks are given in Section 7. The norm kernel of the 3α system defined in the Fock-Bargmann space is given in an explicit form in Appendix A.

2. Theoretical model

The RGM wave function of three-cluster system is represented as an antisymmetrized product of internal cluster functions and wave function of the relative motion of clusters. The latter is found by solving the integro-differential equation, which is obtained from the Schrödinger equation upon integrating over single-particle coordinates. The internal cluster functions are considered to be fixed. Expanding the wave function of cluster relative motion into the basis of the Pauli-allowed multi-particle harmonic oscillator states $\Psi_n(\{\mathbf{r}\})$, one can reduce the integro-differential equation to the set of linear algebraic equations for the expansion coefficients. Further simplification can be achieved by transition from the coordinate space to the space of complex generator parameters (Fock-Bargmann space [15]), because the Fock-Bargmann image $\Psi_n(\{\mathbf{R}\})$ of the Pauli-allowed state $\Psi_n(\{\mathbf{r}\})$ is much simpler than the original. Here, $\{\mathbf{r}\}$ is a set of nucleon position vectors, $\{\mathbf{R}\}$ is a set of generator parameters with the help of which the dynamics of the considered degrees of freedom will be reproduced, and n is a set of quantum numbers of basis functions.

The complete basis of the Pauli-allowed harmonic oscillator states defined both in the coordinate space and in the Fock-Bargmann space is generated by the Slater determinant $\Phi(\{\mathbf{R}\},\{\mathbf{r}\})$ composed of the single-particle orbitals. It is advantageous to use the modified Bloch-Brink orbitals as spatial wave functions of the nucleons, because each orbital serves as the kernel of the transformation from the coordinate representation to the Fock-Bargmann representation and, at the same time, it is generating function of the basis of a harmonic oscillator. For three-cluster systems composed of three s-clusters containing $A_k \leq 4$ nucleons, only orbitals of the following form are required:

$$\phi_k(\mathbf{r}_i) = \frac{1}{\pi^{3/4}} \exp\left(-\frac{1}{2}\mathbf{r}_i^2 + \sqrt{2}(\mathbf{R}_k \cdot \mathbf{r}_i) - \frac{1}{2}\mathbf{R}_k^2\right), \quad i \in A_k,$$

where complex generator parameters $\mathbf{R}_k$ are independent variables of the wave function $\Psi_n(\{\mathbf{R}_k\})$, which describe the positions of the clusters in the Fock-Bargmann space:

$$\mathbf{R}_k = \frac{\vec{\xi}_k + i\vec{\eta}_k}{\sqrt{2}}.$$

$\vec{\xi}_k$ and $\vec{\eta}_k$ are vectors of coordinate and momentum, respectively.

A complete basis of Pauli-allowed states in the Fock-Bargmann space and their eigenvalues can be obtained from the overlap integral $\bar{I}(\mathbf{S}_k, \mathbf{R}_k)$ of the two Slater determinants, which are usually defined as follows:

$$\bar{I}(\mathbf{S}_k, \mathbf{R}_k) = \int \Phi(\mathbf{S}_k, \{\mathbf{r}\})\Phi(\mathbf{R}_k, \{\mathbf{r}\})d\tau.$$

Here, integration is performed over all single-particle variables. The motion of the center of mass is eliminated by transition from the vectors $\mathbf{R}_k$ to the Jacobi vectors

$$\mathbf{R}_{cm} = \frac{1}{\sqrt{A}}\left(A_1\mathbf{R}_1 + A_2\mathbf{R}_2 + A_3\mathbf{R}_3\right),$$

$$\mathbf{a} = \sqrt{\frac{A_1(A_2+A_3)}{A}}\left(\mathbf{R}_1 - \frac{A_2\mathbf{R}_2 + A_3\mathbf{R}_3}{A_2+A_3}\right), \quad \mathbf{b} = \sqrt{\frac{A_2A_3}{A_2+A_3}}(\mathbf{R}_2 - \mathbf{R}_3).$$

Then

$$\bar{I}(\mathbf{S}_k, \mathbf{R}_k) = \exp(\mathbf{R}_{\rm cm}\cdot\mathbf{S}_{\rm cm})\, I(\mathbf{a}, \mathbf{b}; \tilde{\mathbf{a}}, \tilde{\mathbf{b}}),$$

where $I(\mathbf{a}, \mathbf{b}; \tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ is a translation-invariant overlap integral generally known as the norm kernel.

The Pauli-allowed basis states $\Psi_n(\mathbf{a}, \mathbf{b})$ are the eigenfunctions of the norm kernel:

$$\Lambda_n\Psi_n(\mathbf{a}, \mathbf{b}) = \int I(\mathbf{a}, \mathbf{b}; \tilde{\mathbf{a}}, \tilde{\mathbf{b}})\Psi_n(\tilde{\mathbf{a}}^*, \tilde{\mathbf{b}}^*)d\mu_{\tilde{\mathbf{a}},\tilde{\mathbf{b}}},$$

while Λ_n are its eigenvalues. The Bargmann measure $d\mu_{\mathbf{a},\mathbf{b}}$ is defined as follows:

$$d\mu_{\mathbf{a},\mathbf{b}} = \exp\{-(\mathbf{a}\mathbf{a}^*)\}\,\frac{d\vec{\xi}_a d\vec{\eta}_a}{(2\pi)^3}\,\exp\{-(\mathbf{b}\mathbf{b}^*)\}\,\frac{d\vec{\xi}_b d\vec{\eta}_b}{(2\pi)^3}.$$

The Hilbert-Schmidt expansion of the norm kernel

$$I(\mathbf{a}, \mathbf{b}; \tilde{\mathbf{a}}, \tilde{\mathbf{b}}) = \sum_n \Lambda_n\Psi_n(\mathbf{a}, \mathbf{b})\Psi_n(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$$

can be interpreted also as density matrix of mixed system [14], with Λ_n being the elements of the diagonalized density matrix, which are proportional to the realization probability of the system states defined by the corresponding eigenfunctions.

Examine next the explicit expressions for the norm kernels of those three-cluster systems which comprised of three identical s-clusters, such as ${}^3n = n+n+n$, three-boson system, ${}^6n = {}^2n+{}^2n+{}^2n$ and ${}^{12}\mathrm{C}=\alpha+\alpha+\alpha$.

3. Norm kernel of three-fermion system

In this section we consider norm kernel of three identical fermions (for example, three nucleons with the same spin and isospin projections), which is the main building block for the norm kernels of three-cluster systems composed of three identical clusters.

Instead of Jacobi vectors $\mathbf{a}$ and $\mathbf{b}$ let us introduce new complex vectors

$$\mathbf{A} = \frac{\mathbf{a} + i\,\mathbf{b}}{\sqrt{2}}, \quad \mathbf{B} = \frac{\mathbf{a} - i\,\mathbf{b}}{\sqrt{2}}.$$

Similar transformation of creation and annihilation operator has been shown to simplify essentially construction of a complete bispherical harmonic oscillator basis in three-nucleon system [16]. Further we shall demonstrate that such transformation facilitates also construction of SU(3)-basis for three-fermion system and, more importantly, solving eigenvalue and eigenfunction problem for the three-cluster norm kernel.

In terms of vectors $\mathbf{A}$ and $\mathbf{B}$ the norm kernel for three identical nucleons has the following form:

$$I_{n+n+n}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}) = \frac{1}{3!}\exp\left\{(\mathbf{A}\tilde{\mathbf{A}})+(\mathbf{B}\tilde{\mathbf{B}})\right\} + \frac{1}{3!}\exp\left\{e^{i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{A}})+e^{-i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{B}})\right\} +$$
$$+\frac{1}{3!}\exp\left\{e^{-i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{A}})+e^{i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{B}})\right\} - \frac{1}{3!}\exp\left\{(\mathbf{B}\tilde{\mathbf{A}})+(\mathbf{A}\tilde{\mathbf{B}})\right\} -$$
$$-\frac{1}{3!}\exp\left\{e^{i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{A}})+e^{-i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{B}})\right\} - \frac{1}{3!}\exp\left\{e^{-i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{A}})+e^{i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{B}})\right\}.$$

The latter expression can be considered as the result of action of the antisymmetrization operator $\hat{A}$ on the first exponent, which contains a compete basis of both the Pauli-allowed and Pauli-forbidden harmonic oscillator states:

$$I_{n+n+n}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}) = \frac{1}{3!}\hat{A}\exp\left\{(\mathbf{A}\tilde{\mathbf{A}})+(\mathbf{B}\tilde{\mathbf{B}})\right\},$$

where $\hat{A}$ is defined in the Fock-Bargmann space as

$$\hat{A}f(\mathbf{A},\mathbf{B}) = f(\mathbf{A},\mathbf{B}) + f\left(e^{i\frac{2\pi}{3}}\mathbf{A}, e^{-i\frac{2\pi}{3}}\mathbf{B}\right) + f\left(e^{-i\frac{2\pi}{3}}\mathbf{A}, e^{i\frac{2\pi}{3}}\mathbf{B}\right) -$$
$$-f(\mathbf{B},\mathbf{A}) - f\left(e^{i\frac{2\pi}{3}}\mathbf{B}, e^{-i\frac{2\pi}{3}}\mathbf{A}\right) - f\left(e^{-i\frac{2\pi}{3}}\mathbf{B}, e^{i\frac{2\pi}{3}}\mathbf{A}\right). \quad (1)$$

First term corresponds to the identity permutation, second and third terms describe two cyclic permutations, while the last three items are associated with pair permutations and, hence, have the minus sign. So, permutations in the system of three identical nucleons correspond to the rotation by 120 degrees in the plane going through all nucleons. Evidently, such rotation relates to the transition from one Jacobi tree to another:

$$\mathbf{A}_1 = e^{i\frac{2\pi}{3}}\mathbf{B};\quad \mathbf{B}_1 = e^{-i\frac{2\pi}{3}}\mathbf{A}; \mathbf{A}_2 = e^{-i\frac{2\pi}{3}}\mathbf{B};\quad \mathbf{B}_2 = e^{i\frac{2\pi}{3}}\mathbf{A},$$

where

$$\mathbf{A}_{1,2} = \frac{\mathbf{a}_{1,2} + i\,\mathbf{b}_{1,2}}{\sqrt{2}},\ \ \mathbf{B}_{1,2} = \frac{\mathbf{a}_{1,2} - i\,\mathbf{b}_{1,2}}{\sqrt{2}}.$$

Vectors $\mathbf{a}_{1,2}$, $\mathbf{b}_{1,2}$ belong to the alternative Jacobi trees.

Expanding the norm kernel $I_{n+n+n}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)$ in powers of vectors $\mathbf{A}$ and $\mathbf{B}$ we obtain two sums:

$$I_{n+n+n}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = \sum_{m=1}^{\infty}\frac{1}{2(m!)^2}\left\{(\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^m - (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^m\right\} +$$
$$+\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2m!(m+3k)!}\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m+3k}(\mathbf{B}\tilde{\mathbf{B}})^m + (\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^{m+3k} -\right.$$
$$\left.-(\mathbf{B}\tilde{\mathbf{A}})^{m+3k}(\mathbf{A}\tilde{\mathbf{B}})^m - (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^{m+3k}\right\}.$$

Either of two sums generates the states with definite number of oscillator quanta $\bar{\nu}$: the one-fold sum corresponds to $\bar{\nu} = 2m$, while the two-fold sum contains the states with $\bar{\nu} = 2m + 3k$. It should be noticed that only expressions, which are antisymmetric with respect to nucleon permutations, enter the above expansion of the norm kernel.

These sums each are the superpositions of partial norm kernels $i_{(\bar{\lambda},\bar{\mu})}$ possessing definite SU(3)-symmetry and labeled by the SU(3) indices $(\bar{\lambda},\bar{\mu})$ (see Ref. [17]):

$$\frac{1}{2(m!)^2}\left\{(\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^m - (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^m\right\} = \sum_{\bar{\mu}=0}^{m} i_{(2m-2\bar{\mu},\bar{\mu})}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right),$$

$$\frac{1}{2m!(m+3k)!}\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m+3k}(\mathbf{B}\tilde{\mathbf{B}})^m + (\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^{m+3k} - (\mathbf{B}\tilde{\mathbf{A}})^{m+3k}(\mathbf{A}\tilde{\mathbf{B}})^m - (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^{m+3k}\right\} =$$

$$= \sum_{\bar{\mu}=0}^{m} i_{(2m+3k-2\bar{\mu},\bar{\mu})},$$

where

$$i_{(2m-2\bar{\mu},\bar{\mu})}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = \frac{(-1)^{\bar{\mu}}(-m)_{\bar{\mu}}(-m)_{\bar{\mu}}}{2(m!)^2\bar{\mu}!(-2m+\bar{\mu}-1)_{\bar{\mu}}}([\mathbf{AB}][\tilde{\mathbf{A}}\tilde{\mathbf{B}}])^{\bar{\mu}} \times$$
$$\times\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m-\bar{\mu}}(\mathbf{B}\tilde{\mathbf{B}})^{m-\bar{\mu}}{}_2F_1\left(-m+\bar{\mu},-m+\bar{\mu};-2m+2\bar{\mu};z\right) -\right.$$
$$\left.-\,(-1)^{\bar{\mu}}(\mathbf{B}\tilde{\mathbf{A}})^{m-\bar{\mu}}(\mathbf{A}\tilde{\mathbf{B}})^{m-\bar{\mu}}{}_2F_1\left(-m+\bar{\mu},-m+\bar{\mu};-2m+2\bar{\mu};\tilde{z}\right)\right\};$$

$$i_{(2m+3k-2\bar{\mu},\bar{\mu})}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = \frac{(-1)^{\bar{\mu}}(-m)_{\bar{\mu}}(-m-3k)_{\bar{\mu}}}{2m!(m+3k)!\bar{\mu}!(-2m-3k+\bar{\mu}-1)_{\bar{\mu}}}([\mathbf{AB}][\tilde{\mathbf{A}}\tilde{\mathbf{B}}])^{\bar{\mu}} \times$$
$$\times\left\{\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m+3k-\bar{\mu}}(\mathbf{B}\tilde{\mathbf{B}})^{m-\bar{\mu}} + (\mathbf{A}\tilde{\mathbf{A}})^{m-\bar{\mu}}(\mathbf{B}\tilde{\mathbf{B}})^{m+3k-\bar{\mu}}\right\}\times\right.$$
$$\times{}_2F_1\left(-m+\bar{\mu},-m-3k+\bar{\mu};-2m-3k+2\bar{\mu};z\right) -$$
$$-(-1)^{\bar{\mu}}\left\{(\mathbf{B}\tilde{\mathbf{A}})^{m+3k-\bar{\mu}}(\mathbf{A}\tilde{\mathbf{B}})^{m-\bar{\mu}} + (\mathbf{B}\tilde{\mathbf{A}})^{m-\bar{\mu}}(\mathbf{A}\tilde{\mathbf{B}})^{m+3k-\bar{\mu}}\right\}\times$$
$$\left.\times{}_2F_1\left(-m+\bar{\mu},-m-3k+\bar{\mu};-2m-3k+2\bar{\mu};\tilde{z}\right)\right\};$$

Here $(-x)_n$ is the Pochhammer symbol, defined as follows:

$$(-x)_n = (-1)^n\frac{x!}{(x-m)!};$$

${}_2F_1(\alpha,\beta;\gamma;Z)$ is the hypergeometric function with variables

$$z = \frac{([\mathbf{AB}][\tilde{\mathbf{A}}\tilde{\mathbf{B}}])}{(\mathbf{A}\tilde{\mathbf{A}})(\mathbf{B}\tilde{\mathbf{B}})},\quad \tilde{z} = \frac{([\mathbf{BA}][\tilde{\mathbf{A}}\tilde{\mathbf{B}}])}{(\mathbf{B}\tilde{\mathbf{A}})(\mathbf{A}\tilde{\mathbf{B}})}.$$

The number of oscillator quanta $\bar{\nu}$ relates the SU(3)-symmetry indices $(\bar{\lambda},\bar{\mu})$ as $\bar{\nu} = \bar{\lambda}+2\bar{\mu}$.

Partial norm kernels $i_{(\bar{\lambda},\bar{\mu})}$ are normalized to the dimensionality of $(\bar{\lambda},\bar{\mu})$ SU(3)-representation:

$$\int i_{(\bar{\lambda},\bar{\mu})}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}})d\mu_{\mathbf{A},\mathbf{B}} = \frac{(\bar{\lambda}+1)(\bar{\mu}+1)(\bar{\lambda}+\bar{\mu}+2)}{2}. \tag{2}$$

It is remarkable that $i_{(2m-4\mu,2\mu)} = 0$. Hence, the latter representation contains only the Pauli-forbidden states. Evidently, the state corresponding to zero number of oscillator quanta also can not be realized, i.e., $i_{(0,0)} = 0$.

Basis functions characterized by zero total orbital momentum $L = 0$ are generated by the $i_{(2m+3k-2\bar{\mu},\bar{\mu})}$ representations, provided that both SU(3)-indices are even: $2m+3k = 2\nu$, $\bar{\mu} = 2\mu$. Explicitly, these functions are expressible in terms of hypergeometric functions ${}_2F_1(\alpha,\beta;\gamma;Z)$, with the variable

$$Z = \frac{[\mathbf{AB}]^2}{\mathbf{A}^2\mathbf{B}^2}.$$

Namely,

$$\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}) = N^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}\left(\mathbf{A}^{6k}-\mathbf{B}^{6k}\right)\mathbf{A}^{2n-2\mu}\mathbf{B}^{2n-2\mu}[\mathbf{AB}]^{2\mu}\times$$
$$\times\begin{cases}{}_2F_1\left(-\nu+n+\mu,-n+\mu;-\nu+2\mu+\frac{1}{2};Z\right), & \nu-3k=2n;\\ (\mathbf{AB})\,{}_2F_1\left(-\nu+n+1+\mu,-n+\mu;-\nu+2\mu+\frac{1}{2};Z\right), & \nu-3k=2n+1,\end{cases}$$

$$N^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)} = \frac{1}{\sqrt{2(\nu+3k-2\mu)!(\nu-3k-2\mu)!}} \times$$
$$\times \sqrt{\frac{(2\nu+1-4\mu)!}{(2\mu)!(2\nu+1-2\mu)!}} \sqrt{\frac{(2\nu-2\mu)!!(2\nu-4\mu-1)!!(2\mu-1)!!}{(2\nu-2\mu+1)!!(2\nu-4\mu)!!(2\mu)!!}}.$$

A set of quantum numbers of functions $\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ includes the total number of oscillator quanta 2ν, the SU(3) symmetry indices (λ,μ), and additional quantum number k for the SU(3)-degenerated states [1]. Total orbital momentum L and its projection M are equal to zero and, hence, will be omitted in what follows.

Functions $\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ comply with requirements of the Pauli exclusion principle. It is easy to verify this acting on these functions by antisymmetrization operator $\hat{A}$ defined in (1).

$$\hat{A}\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}) = \Lambda_{(2\nu-4\mu,2\mu)_k}\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}, \quad \Lambda_{(2\nu-4\mu,2\mu)_k} = \begin{cases} 1,\ 1 \le k \le \left[\frac{\nu-2\mu}{3}\right]; \\ 0,\ k=0. \end{cases}$$

Here $[x]$ denotes an integer part of the number x.

Hence, at each value of the number of oscillator quanta ν there are $\left[\frac{\nu-2\mu}{3}\right]$ Pauli-allowed basis functions and one Pauli-forbidden state belonging to the same SU(3) representation $(2\nu-4\mu,2\mu)$. All the Pauli-allowed states have unit eigenvalues, while the Pauli-forbidden states correspond to zero eigenvalues. This implies that a norm kernel of three-nucleon system can be treated as the density matrix of the pure system. Indeed, in this case dynamics of all degrees of freedom is considered and no averaging is performed.

In representation of $\mathbf{A}$ and $\mathbf{B}$ vectors an SU(3)-projected norm kernel $i_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}})$ has a diagonal form:

$$i_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}) = \sum_{k=1}^{\left[\frac{\nu-2\mu}{3}\right]} \chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\tilde{\mathbf{A}},\tilde{\mathbf{B}}).$$

Hence, solving the eigenvalue and eigenvectors problem for three-nucleon system becomes a trivial procedure. It is also easy to distinguish basis states belonging to the SU(3)-degenerated representation by the additional quantum number k, which explicitly enters the expression for the basis functions. However, due to the fact that all eigenvalues of the Pauli-allowed states equal to unity, eigenfunctions of the norm kernel are not uniquely determined. Any unitary transformation applied to the functions $\chi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ gives a new set of basis functions, which are also antisymmetric with respect to the nucleon permutation. For example, hyperspherical harmonics can be successfully applied to studying three-nucleon systems.

4. Norm kernel of three-boson system

In this section we consider norm kernel of three identical bosons $I_{3b}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}})$, which generates basis functions needed for the description of three-cluster systems composed of three identical boson clusters, such as ${}^6n = {}^2n + {}^2n + {}^2n$ and ${}^{12}\mathrm{C} = \alpha+\alpha+\alpha$.

In terms of vectors $\mathbf{A}$ and $\mathbf{B}$ the norm kernel for three identical bosons is very similar to that of three fermions:

$$I_{3b}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}) = \frac{1}{3!}\exp\left\{(\mathbf{A}\tilde{\mathbf{A}})+(\mathbf{B}\tilde{\mathbf{B}})\right\} + \frac{1}{3!}\exp\left\{e^{i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{A}})+e^{-i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{B}})\right\} +$$
$$+ \frac{1}{3!}\exp\left\{e^{-i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{A}})+e^{i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{B}})\right\} + \frac{1}{3!}\exp\left\{(\mathbf{B}\tilde{\mathbf{A}})+(\mathbf{A}\tilde{\mathbf{B}})\right\} +$$
$$+ \frac{1}{3!}\exp\left\{e^{i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{A}})+e^{-i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{B}})\right\} + \frac{1}{3!}\exp\left\{e^{-i\frac{2\pi}{3}}(\mathbf{B}\tilde{\mathbf{A}})+e^{i\frac{2\pi}{3}}(\mathbf{A}\tilde{\mathbf{B}})\right\}.$$

[1] An upper index of basis functions corresponds to the minimal of the number of quanta along $\mathbf{A}$ and $\mathbf{B}$

The only difference between the norm kernels of three fermions and three bosons is that the latter is symmetric with respect to nucleon permutation, while the former is antisymmetric. Naturally, $I_{3b}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}})$ can be considered as the result of action of the symmetrization operator $\hat{P}$ on the first exponent:

$$I_{3b}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}) = \frac{1}{3!}\hat{P}\exp\left\{(\mathbf{A}\tilde{\mathbf{A}}) + (\mathbf{B}\tilde{\mathbf{B}})\right\},$$

where $\hat{P}$ is defined in the Fock-Bargmann space as

$$\begin{aligned}\hat{P}f(\mathbf{A},\mathbf{B}) = f(\mathbf{A},\mathbf{B}) + f\left(e^{i\frac{2\pi}{3}}\mathbf{A}, e^{-i\frac{2\pi}{3}}\mathbf{B}\right) + f\left(e^{-i\frac{2\pi}{3}}\mathbf{A}, e^{i\frac{2\pi}{3}}\mathbf{B}\right) + \\ + f(\mathbf{B},\mathbf{A}) + f\left(e^{i\frac{2\pi}{3}}\mathbf{B}, e^{-i\frac{2\pi}{3}}\mathbf{A}\right) + f\left(e^{-i\frac{2\pi}{3}}\mathbf{B}, e^{i\frac{2\pi}{3}}\mathbf{A}\right). \qquad (3)\end{aligned}$$

Meaning of each term is the same as in Eq. (1).

Expansion of the norm kernel $I_{3b}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)$ in powers of vectors $\mathbf{A}$ and $\mathbf{B}$ gives the result:

$$\begin{aligned}I_{3b}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = \sum_{m=1}^{\infty}\frac{1}{2(m!)^2}\left\{(\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^m + (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^m\right\} + \\ + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2m!(m+3k)!}\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m+3k}(\mathbf{B}\tilde{\mathbf{B}})^m + (\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^{m+3k} + \right. \\ \left. + (\mathbf{B}\tilde{\mathbf{A}})^{m+3k}(\mathbf{A}\tilde{\mathbf{B}})^m + (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^{m+3k}\right\}.\end{aligned}$$

Of course, only expressions, which are symmetric with respect to boson permutation, enter the above expansion of the norm kernel. Notice, that the above-mentioned symmetry results in essential reduction of the basis states, which can be realized in three-boson norm kernel. Indeed, the first exponent $\exp\left\{(\mathbf{A}\tilde{\mathbf{A}}) + (\mathbf{B}\tilde{\mathbf{B}})\right\}$, which generates both the Pauli-allowed and the Pauli-forbidden basis functions, contains six times more states than $I_{3b}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)$ norm kernel.

By analogy with three-fermion system, each sum corresponds to the fixed number of quanta $\bar{\nu}$ and can be presented as a sum of partial norm kernels $\bar{i}_{(\bar{\lambda},\bar{\mu})}$ belonging to the SU(3)-representations $(\bar{\lambda},\bar{\mu})$:

$$\frac{1}{2(m!)^2}\left\{(\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^m + (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^m\right\} = \sum_{\bar{\mu}=0}^{m}\bar{i}_{(2m-2\bar{\mu},\bar{\mu})}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right),$$

$$\begin{aligned}\frac{1}{2m!(m+3k)!}\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m+3k}(\mathbf{B}\tilde{\mathbf{B}})^m + (\mathbf{A}\tilde{\mathbf{A}})^m(\mathbf{B}\tilde{\mathbf{B}})^{m+3k} + (\mathbf{B}\tilde{\mathbf{A}})^{m+3k}(\mathbf{A}\tilde{\mathbf{B}})^m + (\mathbf{B}\tilde{\mathbf{A}})^m(\mathbf{A}\tilde{\mathbf{B}})^{m+3k}\right\} = \\ = \sum_{\bar{\mu}=0}^{m}\bar{i}_{(2m+3k-2\bar{\mu},\bar{\mu})},\end{aligned}$$

where

$$\begin{aligned}\bar{i}_{(2m-2\bar{\mu},\bar{\mu})}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = \frac{(-1)^{\bar{\mu}}(-m)_{\bar{\mu}}(-m)_{\bar{\mu}}}{2(m!)^2\bar{\mu}!(-2m+\bar{\mu}-1)_{\bar{\mu}}}([\mathbf{AB}][\tilde{\mathbf{A}}\tilde{\mathbf{B}}])^{\bar{\mu}} \times \\ \times\left\{(\mathbf{A}\tilde{\mathbf{A}})^{m-\bar{\mu}}(\mathbf{B}\tilde{\mathbf{B}})^{m-\bar{\mu}}{}_2F_1\left(-m+\bar{\mu},-m+\bar{\mu};-2m+2\bar{\mu};z\right) + \right. \\ \left. + (-1)^{\bar{\mu}}(\mathbf{B}\tilde{\mathbf{A}})^{m-\bar{\mu}}(\mathbf{A}\tilde{\mathbf{B}})^{m-\bar{\mu}}{}_2F_1\left(-m+\bar{\mu},-m+\bar{\mu};-2m+2\bar{\mu};\tilde{z}\right)\right\};\end{aligned}$$

$$\bar{i}_{(2m+3k-2\bar{\mu},\bar{\mu})}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = \frac{(-1)^{\bar{\mu}}(-m)_{\bar{\mu}}(-m-3k)_{\bar{\mu}}}{2m!(m+3k)!\bar{\mu}!(-2m-3k+\bar{\mu}-1)_{\bar{\mu}}}([\mathbf{AB}][\tilde{\mathbf{A}}\tilde{\mathbf{B}}])^{\bar{\mu}} \times$$

$$\times \left\{ \left\{ (\mathbf{A}\tilde{\mathbf{A}})^{m+3k-\bar{\mu}} (\mathbf{B}\tilde{\mathbf{B}})^{m-\bar{\mu}} + (\mathbf{A}\tilde{\mathbf{A}})^{m-\bar{\mu}} (\mathbf{B}\tilde{\mathbf{B}})^{m+3k-\bar{\mu}} \right\} \times \right.$$
$$\times {}_2F_1\left(-m+\bar{\mu}, -m-3k+\bar{\mu}; -2m-3k+2\bar{\mu}; z\right) +$$
$$+(-1)^{\bar{\mu}} \left\{ (\mathbf{B}\tilde{\mathbf{A}})^{m+3k-\bar{\mu}} (\mathbf{A}\tilde{\mathbf{B}})^{m-\bar{\mu}} + (\mathbf{B}\tilde{\mathbf{A}})^{m-\bar{\mu}} (\mathbf{A}\tilde{\mathbf{B}})^{m+3k-\bar{\mu}} \right\} \times$$
$$\left. \times {}_2F_1\left(-m+\bar{\mu}, -m-3k+\bar{\mu}; -2m-3k+2\bar{\mu}; \tilde{z}\right) \right\}.$$

Arguments z and $\tilde{z}$ of the hypergeometric functions are the same as those in the previous section.

In contrast to three-fermion system, $\bar{i}_{(2m-4\mu,2\mu)} \neq 0$, while $\bar{i}_{(2m-4\mu-2,2\mu+1)} = 0$. Therefore, for three-boson system all the states belonging to $(2\nu-4\mu,2\mu)$ representation are allowed by the Pauli principle. Basis functions characterized by zero total orbital momentum $L=0$ are of the following form:

$$\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}) = \bar{N}^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)} \left(\mathbf{A}^{6k} + \mathbf{B}^{6k}\right) \mathbf{A}^{2n-2\mu}\mathbf{B}^{2n-2\mu}[\mathbf{AB}]^{2\mu} \times$$
$$\times \begin{cases} {}_2F_1\left(-\nu+n+\mu, -n+\mu; -\nu+2\mu+\frac{1}{2}; Z\right), & \nu-3k=2n; \\ (\mathbf{AB})\, {}_2F_1\left(-\nu+n+1+\mu, -n+\mu; -\nu+2\mu+\frac{1}{2}; Z\right), & \nu-3k=2n+1, \end{cases} \tag{4}$$

$$\bar{N}^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)} = \frac{1}{\sqrt{2(1+\delta_{k,0})(\nu+3k-2\mu)!(\nu-3k-2\mu)!}} \times$$
$$\times \sqrt{\frac{(2\nu+1-4\mu)!}{(2\mu)!(2\nu+1-2\mu)!}} \sqrt{\frac{(2\nu-2\mu)!!(2\nu-4\mu-1)!!(2\mu-1)!!}{(2\nu-2\mu+1)!!(2\nu-4\mu)!!(2\mu)!!}}.$$

Functions $\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ are the eigenfunctions of the symmetrization operator $\hat{P}$ defined in (3):

$$\hat{P}\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}) = \psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}, \quad 0 \le k \le \left[\frac{\nu-2\mu}{3}\right].$$

Finally, at each value of the number of oscillator quanta $\left[\frac{\nu-2\mu}{3}\right]+1$ basis functions, belonging to SU(3)-representation $(2\nu-4\mu,2\mu)$ and having zero orbital momentum, can be constructed, and all these states are consistent with the requirements of the Pauli principle.

In the next section we will demonstrate that the Pauli-allowed states of three-cluster systems composed of identical boson clusters can be represented as a superposition of eigenfunctions $\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ of three-boson norm kernel. Indeed, the structure of the latter functions reflects the symmetry with respect to the permutation of clusters as a whole and, in addition, these functions are the eigenfunctions of the second-order Casimir operator. Further it will be shown that using $\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ functions we essentially simplify the eigenvalue and eigenfunction problem for three-cluster systems.

5. Norm kernel of three dineutrons

Norm kernel of three-dineutron system $I_{3d'}$ can be obtained by squaring of the three-neutron norm kernel I_{3n}:

$$I_{3d'}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right) = 3!\left[I_{3n}\left(\frac{\mathbf{A}}{\sqrt{2}},\frac{\mathbf{B}}{\sqrt{2}};\frac{\tilde{\mathbf{A}}}{\sqrt{2}},\frac{\tilde{\mathbf{B}}}{\sqrt{2}}\right)\right]^2.$$

Replacement

$$\mathbf{A} \to \frac{1}{\sqrt{2}}\mathbf{A}, \quad \mathbf{B} \to \frac{1}{\sqrt{2}}\mathbf{B}, \quad \tilde{\mathbf{A}} \to \frac{1}{\sqrt{2}}\tilde{\mathbf{A}}, \quad \tilde{\mathbf{B}} \to \frac{1}{\sqrt{2}}\tilde{\mathbf{B}}$$

is required to ensure correct elimination of the center-of-mass motion in $3d'$ system. Such scaling transformation of the Jacobi vectors accounts for the fact that each cluster in the $3d'$ system consists of two neutrons.

The norm kernel $I_{3d'}$ can be presented as a three-termed antisymmetrization operator acting on the exponent, which contains a complete basis of both the Pauli-allowed and the Pauli-forbidden states:

$$I_{3d'}(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}) = \frac{1}{3!}(\hat{A}_1+\hat{A}_2+\hat{A}_3)\exp\left\{(\mathbf{A}\tilde{\mathbf{A}})+(\mathbf{B}\tilde{\mathbf{B}})\right\}.$$

Each of three above-mentioned operators combines certain group of permutations.

The first operator $\hat{A}_1$ coincides with the symmetrization operator $\hat{P}$ of three-boson system defined in (3). Hence, $\hat{A}_1 = \hat{P}$ corresponds to the permutations of dineutron clusters as a whole.

The second part of antisymmetrization operator is defined as

$$\begin{aligned}\hat{A}_2 f(\mathbf{A},\mathbf{B}) = 2\Bigg\{ & f\left(-\frac{\mathbf{A}}{2},-\frac{\mathbf{B}}{2}\right) + f\left(-e^{i\frac{2\pi}{3}}\frac{\mathbf{A}}{2},-e^{-i\frac{2\pi}{3}}\frac{\mathbf{B}}{2}\right) + f\left(-e^{-i\frac{2\pi}{3}}\frac{\mathbf{A}}{2},-e^{i\frac{2\pi}{3}}\frac{\mathbf{B}}{2}\right) + \\ & + f\left(-\frac{\mathbf{B}}{2},-\frac{\mathbf{A}}{2}\right) + f\left(-e^{i\frac{2\pi}{3}}\frac{\mathbf{B}}{2},-e^{-i\frac{2\pi}{3}}\frac{\mathbf{A}}{2}\right) + f\left(-e^{-i\frac{2\pi}{3}}\frac{\mathbf{B}}{2},-e^{i\frac{2\pi}{3}}\frac{\mathbf{A}}{2}\right)\Bigg\}. \end{aligned} \tag{5}$$

It is associated with permutations of nucleons between all three dineutron clusters and includes simultaneous pair or cyclic permutations of spin-up and spin-down neutrons. Notice that only the first term of operator $\hat{A}_2$ results in a rearrangement of clusters due to the permutation of identical neutrons. Remaining five terms can be obtained by permutation of clusters as a whole following the rearrangement associated with the first term.

Finally, the last operator $\hat{A}_3$ is a little bit more complicated:

$$\hat{A}_3 = \hat{A}_3^0 + \hat{A}_3^+ + \hat{A}_3^-,$$

where

$$\begin{aligned}\hat{A}_3^0 f(\mathbf{A},\mathbf{B}) = -2\Bigg\{ & f\left(\frac{\mathbf{A}+\mathbf{B}}{2},\frac{\mathbf{A}+\mathbf{B}}{2}\right) + f\left(e^{i\frac{2\pi}{3}}\frac{\mathbf{A}+\mathbf{B}}{2},e^{-i\frac{2\pi}{3}}\frac{\mathbf{A}+\mathbf{B}}{2}\right) + \\ & + f\left(e^{-i\frac{2\pi}{3}}\frac{\mathbf{A}+\mathbf{B}}{2},e^{i\frac{2\pi}{3}}\frac{\mathbf{A}+\mathbf{B}}{2}\right)\Bigg\}; \\ \hat{A}_3^+ f(\mathbf{A},\mathbf{B}) = -2\Bigg\{ & f\left(\frac{1}{2}\left[e^{i\frac{2\pi}{3}}\mathbf{A}+e^{-i\frac{2\pi}{3}}\mathbf{B}\right],\frac{1}{2}\left[e^{i\frac{2\pi}{3}}\mathbf{A}+e^{-i\frac{2\pi}{3}}\mathbf{B}\right]\right) \\ & + f\left(\frac{e^{i\frac{2\pi}{3}}}{2}\left[e^{i\frac{2\pi}{3}}\mathbf{A}+e^{-i\frac{2\pi}{3}}\mathbf{B}\right],\frac{e^{-i\frac{2\pi}{3}}}{2}\left[e^{i\frac{2\pi}{3}}\mathbf{A}+e^{-i\frac{2\pi}{3}}\mathbf{B}\right]\right) + \\ & + f\left(\frac{e^{-i\frac{2\pi}{3}}}{2}\left[e^{i\frac{2\pi}{3}}\mathbf{A}+e^{-i\frac{2\pi}{3}}\mathbf{B}\right],\frac{e^{i\frac{2\pi}{3}}}{2}\left[e^{i\frac{2\pi}{3}}\mathbf{A}+e^{-i\frac{2\pi}{3}}\mathbf{B}\right]\right)\Bigg\}; \\ \hat{A}_3^- f(\mathbf{A},\mathbf{B}) = -2\Bigg\{ & f\left(\frac{1}{2}\left[e^{-i\frac{2\pi}{3}}\mathbf{A}+e^{i\frac{2\pi}{3}}\mathbf{B}\right],\frac{1}{2}\left[e^{-i\frac{2\pi}{3}}\mathbf{A}+e^{i\frac{2\pi}{3}}\mathbf{B}\right]\right) + \\ & + f\left(\frac{e^{i\frac{2\pi}{3}}}{2}\left[e^{-i\frac{2\pi}{3}}\mathbf{A}+e^{i\frac{2\pi}{3}}\mathbf{B}\right],\frac{e^{-i\frac{2\pi}{3}}}{2}\left[e^{-i\frac{2\pi}{3}}\mathbf{A}+e^{i\frac{2\pi}{3}}\mathbf{B}\right]\right) + \\ & + f\left(\frac{e^{-i\frac{2\pi}{3}}}{2}\left[e^{-i\frac{2\pi}{3}}\mathbf{A}+e^{i\frac{2\pi}{3}}\mathbf{B}\right],\frac{e^{i\frac{2\pi}{3}}}{2}\left[e^{-i\frac{2\pi}{3}}\mathbf{A}+e^{i\frac{2\pi}{3}}\mathbf{B}\right]\right)\Bigg\}. \end{aligned} \tag{6}$$

Operator $\hat{A}_3$ combines permutations of nucleons between two clusters, with the third cluster being a spectator. In this case the spin-up neutrons undergo cyclic permutation, while the spin-down neutrons experience pair permutation and vice versa. Depending on which cluster is a spectator, we obtain operators $\hat{A}_3^0$, $\hat{A}_3^+$ or $\hat{A}_3^-$. The three-termed structure of each operator corresponds to three cyclic permutations of clusters as a whole.

It is natural to seek the eigenfunctions of the $3d'$ norm kernel in the form of superposition of the eigenfunctions of the three-boson norm kernel, because the dineutron clusters should obey boson statistics. Hence,

the Pauli-allowed basis functions of the three-dineutron system $\Psi_{(2\nu-4\mu,2\mu)_k}(\mathbf{A},\mathbf{B})$ characterized by positive parity, zero values of total orbital momentum and spin $L=S=0$ and belonging to the $(2\nu-4\mu,2\mu)$ representation of the SU(3)-group can be written as:

$$\Psi_{(2\nu-4\mu,2\mu)_k}(\mathbf{A},\mathbf{B})=\sum_{q=0}^{[\frac{\nu-2\mu}{3}]} D^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)_k}\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}). \tag{7}$$

Expansion coefficients $D^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)_k}$, along with eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$ corresponding to the Pauli-allowed states $\Psi_{(2\nu-4\mu,2\mu)_k}(\mathbf{A},\mathbf{B})$, can be found by diagonalization of the three-dineutron system norm kernel $i^{3d'}_{(2\nu-4\mu,2\mu)}$ projected on the states with the fixed number of quanta and definite SU(3)-symmetry:

$$\begin{aligned} i^{3d'}_{(2\nu-4\mu,2\mu)}=&\sum_{q=0}^{[\frac{\nu-2\mu}{3}]}\sum_{\tilde{q}=0}^{[\frac{\nu-2\mu}{3}]-q}\langle 2\nu,\nu-3q-2\mu|2\nu,\nu-3q-3\tilde{q}-2\mu\rangle_{3d'}\times\\ &\times\left(\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\tilde{\psi}^{\nu-3q-3\tilde{q}-2\mu}_{(2\nu-4\mu,2\mu)}+\psi^{\nu-3q-3\tilde{q}-2\mu}_{(2\nu-4\mu,2\mu)}\tilde{\psi}^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\right). \end{aligned} \tag{8}$$

In order to construct characteristic equation, let us consider the action of the antisymmetrization operator $\hat{A}=\hat{A}_1+\hat{A}_2+\hat{A}_3$ on the functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}$.

It is easy to verify that

$$\frac{1}{3!}\hat{A}_1\,\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})=\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}),\quad \frac{1}{3!}\hat{A}_2\,\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})=\frac{1}{2^{2\nu-1}}\,\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}).$$

Notice that all terms entering the expressions specifying operators $\hat{A}_1$ or $\hat{A}_2$ give the same result acting on basis functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}$.

The last part of antisymmetrization operator provides nonzero result only for $(2\nu,0)$ representation:

$$\begin{aligned} \frac{1}{3!}\hat{A}_3\,\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})=&-\delta_{\mu,0}\frac{6}{4^\nu}\sum_{\tilde{q}=0}^{[\nu/3]-q}\psi^{\nu-3q-3\tilde{q}-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})\,\frac{1}{\sqrt{(1+\delta_{q,0})(1+\delta_{\tilde{q},-q})}}\times\\ &\times\frac{(2\nu)!}{\sqrt{(\nu+3q+3\tilde{q})!(\nu-3q-3\tilde{q})!(\nu+3q)!(\nu-3q)!}}. \end{aligned}$$

Consequently, the matrix elements of the secular equation look like

$$\begin{aligned} \langle 2\nu,\nu-3q-2\mu|2\nu,\nu-3q-3\tilde{q}-2\mu\rangle_{3d'}=&\left(1+\frac{1}{2^{2\nu-1}}\right)\delta_{\tilde{q},0}-\delta_{\mu,0}\frac{6}{4^\nu}\frac{1}{\sqrt{(1+\delta_{q,0})(1+\delta_{\tilde{q},-q})}}\times\\ &\times\frac{(2\nu)!}{\sqrt{(\nu+3q+3\tilde{q})!(\nu-3q-3\tilde{q})!(\nu+3q)!(\nu-3q)!}}. \end{aligned}$$

The action of the antisymmetrization operator $\hat{A}$ on the basis functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}$ results only in diagonal matrix elements of the characteristic equation for $\mu\neq 0$:

$$\hat{A}\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})=\left(1+\frac{1}{2^{2\nu-1}}\right)\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}),\quad \mu\neq 0.$$

By this reason, the Pauli-allowed basis functions of the $3d'$ system belonging to the SU(3) representation $(2\nu-4\mu,2\mu)$ with $\mu\neq 0$ coincide with the eigenfunctions of the three-boson norm kernel and have identical, although not equal to unity, eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$:

$$\Psi_{(2\nu-4\mu,2\mu)_k}(\mathbf{A},\mathbf{B})=\psi^{\nu-3k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}),\quad \Lambda_{(2\nu-4\mu,2\mu)_k}=1+\frac{1}{2^{2\nu-1}},\quad 0\leq k\leq\left[\frac{\nu-2\mu}{3}\right],\quad \mu\neq 0.$$

The Pauli-allowed states of $(2\nu,0)$ representations are characterized by the same eigenvalues at a given number of oscillator quanta ν. However, there is also one Pauli-forbidden state, corresponding to zero eigenvalue:

$$\Lambda_{(2\nu,0)} = (1-\delta_{k,0})\left(1+\frac{1}{2^{2\nu-1}}\right), \quad 0 \le k \le \left[\frac{\nu}{3}\right].$$

Eigenfunctions $\Psi_{(2\nu,0)_k}(\mathbf{A},\mathbf{B})$ of the antisymmetrization operator have more complicated form, being a superposition of functions $\psi^{\nu-3q}_{(2\nu,0)}$ (see Eq. (7)). However, with increasing the number of oscillator quanta the Pauli-forbidden state $\Psi_{(2\nu,0)_{k=0}}(\mathbf{A},\mathbf{B})$ takes simple analytical form:

$$\Psi_{(2\nu,0)_{k=0}} \to \frac{1}{\sqrt{3}}\frac{1}{\sqrt{(2\nu+1)!}}\left(\mathbf{a}^{2\nu}+\mathbf{a}_1^{2\nu}+\mathbf{a}_2^{2\nu}\right) =$$
$$= \sum_{q=0}^{[\nu/3]}\sqrt{\frac{6}{1+\delta_{q,0}}}\left(\frac{1}{2}\right)^{\nu}\sqrt{\frac{(2\nu)!}{(\nu-3q)!(\nu+3q)!}}\psi^{\nu-3q}_{(2\nu,0)}(\mathbf{A},\mathbf{B}).$$

It is quite obvious that such asymptotic behavior corresponds to the case when a dineutron moves away from the ${}^4n = {}^2n + {}^2n$ subsystem (tetraneutron), with the latter being in the Pauli-forbidden state.

Similar behavior is peculiar also for the Pauli-allowed states. By means of the unitary transformation the eigenfunctions of the $3d'$ norm kernel corresponding to nonzero eigenvalues can be rearranged so that

$$\Psi_{(2\nu-4\mu,2\mu)_k} \to \Psi^{\rm as}_{(2\nu-4\mu,2\mu)_k} = \frac{1}{\sqrt{3}}\left(\phi^{2k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{a},\mathbf{b})+\phi^{2k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{a}_1,\mathbf{b}_1)+\phi^{2k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{a}_2,\mathbf{b}_2)\right) =$$
$$= \sum_{q=0}^{[\frac{\nu-2\mu}{3}]}\sqrt{\frac{6}{1+\delta_{q,0}}}\,\bar{\mathcal{K}}^{1/2}_{2k}(\nu-3q-2\mu)\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B}). \qquad (9)$$

Here functions $\phi^{k-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{a},\mathbf{b})$ are defined as

$$\phi^{2m-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{a},\mathbf{b}) = N^{2m-2\mu}_{(2\nu-4\mu,2\mu)}[\mathbf{ab}]^{2\mu}\mathbf{a}^{2\nu-2m-2\mu}\mathbf{b}^{2m-2\mu}\times$$
$$\times {}_2F_1\left(-\nu+m+\mu,-m+\mu;-\nu+2\mu+\frac{1}{2};Z\right), \quad Z=\frac{[\mathbf{ab}]^2}{\mathbf{a}^2\mathbf{b}^2}$$

with $N^{2m-2\mu}_{(2\nu-4\mu,2\mu)}$ being a normalization coefficient:

$$N^{2m-2\mu}_{(2\nu-4\mu,2\mu)} = \sqrt{\frac{(2\nu+1-4\mu)!(2\nu-2\mu)!!(2\nu-4\mu-1)!!(2\mu-1)!!}{(2\mu)!(2\nu+1-2\mu)!(2\nu-2m-2\mu)!(2m-2\mu)!(2\nu-2\mu+1)!!(2\nu-4\mu)!!(2\mu)!!}}.$$

$\bar{\mathcal{K}}^{1/2}_{2k}(\nu-3q-2\mu)$ are the so-called Kravchuk polynomials of a discrete variable [18], which are specified and orthogonal on the interval $2\mu \le m \le 2\nu-2\mu$:

$$\bar{\mathcal{K}}^{1/2}_{2k}(\nu-3q-2\mu) = \left(\frac{1}{2}\right)^{\nu-2\mu}\frac{(2\nu-4\mu)!\,{}_2F_1(-2k+2\mu,-\nu+3q+2\mu;-2\nu+4\mu;2)}{\sqrt{(2k-2\mu)!(2\nu-2k-2\mu)!(\nu-3q-2\mu)!(\nu+3q-2\mu)!}}. \qquad (10)$$

Hence, index of the Kravchuk polynomials serves as additional quantum number for the SU(3)-degenerated states. In Ref. [13] we have shown that a family of eigenfunctions of the antisymmetrization operator combining the states $\Psi_{(2\nu-4\mu,2\mu)_k}$ with the same index k corresponds to a certain binary decay channel of a three-cluster system into a two-cluster subsystem occurring in a ground or an excited harmonic-oscillator state and a remaining cluster. In turn, eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$ of three-cluster system tend to the eigenvalues of a two-cluster subsystem with increasing the number of oscillator quanta ν. However, all eigenvalues of the Pauli-allowed states of ${}^2n + {}^2n$ subsystem are equal to unity, in contrast to the eigenvalues of the ${}^3\mathrm{H}+n$

subsystem of the ^{5}H considered in [13]. By this reason, eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$ of the $3d'$ system are equal and tend to unity for $1 \leq k \leq \left[\frac{\nu-2\mu}{3}\right]$. Hence, strictly speaking, the Pauli-allowed states of the $3d'$ system are not uniquely determined and any unitary transformation can be applied to these states without violating the Pauli principle. It would seem that a basis of hyperspherical harmonics also could be used for study of the $3d'$ system. However, existence of a Pauli-forbidden state makes this statement quite doubtful. Indeed, since a Pauli-forbidden state and the Pauli-allowed states have different eigenvalues, no unitary transformation can leave the diagonal form of the norm kernel intact. Hence, transformation to the hyperspherical basis could result in violation of the Pauli exclusion principle.

6. Norm kernel of three α-particles

Translation-invariant norm kernel for 3α system can be obtained by squaring the norm kernel for three-dineutron system:

$$I_{3\alpha}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)=3!\left[I_{3d'}\left(\frac{\mathbf{A}}{\sqrt{2}},\frac{\mathbf{B}}{\sqrt{2}};\frac{\tilde{\mathbf{A}}}{\sqrt{2}},\frac{\tilde{\mathbf{B}}}{\sqrt{2}}\right)\right]^2\equiv\sum_{i=1}^{6}\frac{1}{3!}\hat{\mathcal{A}}_i\exp\left\{(\mathbf{A}\tilde{\mathbf{A}})+(\mathbf{B}\tilde{\mathbf{B}})\right\}$$

For convenience we divided the antisymmetrization operator $\hat{\mathcal{A}}$ of the 3α-system into six parts $\hat{\mathcal{A}}_i$, each corresponding to a certain group of nucleon permutations [2]. Six parts $I^{(i)}_{3\alpha}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)$ of the norm kernel $I_{3\alpha}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)$, which correspond to constituents $\hat{\mathcal{A}}_i$ of the antisymmetrization operator $\hat{\mathcal{A}}$, are given in an explicit form in Appendix A.

In this section we are interested in the states of positive parity and zero orbital momentum. Hence, all the Pauli-allowed states belong to the $(2\nu-4\mu,2\mu)$ representation of the SU(3) group, like in the case of $3d'$ system. So specifying operators $\hat{\mathcal{A}}_i$ we have taken into account the symmetry peculiar to the basis functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ on which these operators should act. Below we shall explain this statement in detail.

Example of the $3d'$ system illustrates that a permutation of nucleons belonging to different clusters can be reduced to the following mathematical operation:

$$f\left(\mathbf{A},\mathbf{B}\right)\rightarrow const\cdot f\left(\alpha\mathbf{A}+\beta\mathbf{B},\beta^*\mathbf{A}+\alpha^*\mathbf{B}\right), \tag{11}$$

where α,β are some complex numbers. Hence, due to the above-mentioned permutation, some rearrangement of the clusters occurs. Then cyclic permutations of such rearranged clusters can be described as

$$f\left(\mathbf{A},\mathbf{B}\right)\rightarrow const\cdot f\left(e^{\pm i\frac{2\pi}{3}}(\alpha\mathbf{A}+\beta\mathbf{B}),e^{\mp i\frac{2\pi}{3}}(\beta^*\mathbf{A}+\alpha^*\mathbf{B})\right).$$

Obviously, the latter cyclic permutations should give the same result as the initial permutation (11):

$$\begin{aligned}&\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\left(\alpha\mathbf{A}+\beta\mathbf{B},\beta^*\mathbf{A}+\alpha^*\mathbf{B}\right)+\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\left(e^{i\frac{2\pi}{3}}(\alpha\mathbf{A}+\beta\mathbf{B}),e^{-i\frac{2\pi}{3}}(\beta^*\mathbf{A}+\alpha^*\mathbf{B})\right)+\\&+\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\left(e^{-i\frac{2\pi}{3}}(\alpha\mathbf{A}+\beta\mathbf{B}),e^{i\frac{2\pi}{3}}(\beta^*\mathbf{A}+\alpha^*\mathbf{B})\right)=3\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\left(\alpha\mathbf{A}+\beta\mathbf{B},\beta^*\mathbf{A}+\alpha^*\mathbf{B}\right).\end{aligned}$$

In addition, functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$ are symmetric with respect to the interchange of vectors $\mathbf{A}$ and $\mathbf{B}$. [3] By this reason, the following equality holds true:

$$\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\left(\beta^*\mathbf{A}+\alpha^*\mathbf{B},\alpha\mathbf{A}+\beta\mathbf{B}\right)=\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\left(\alpha\mathbf{A}+\beta\mathbf{B},\beta^*\mathbf{A}+\alpha^*\mathbf{B}\right).$$

Keeping in mind all of the preceding, let us determine operators $\hat{\mathcal{A}}_i$. The first part $\hat{\mathcal{A}}_1$ of the antisymmetrization operator $\hat{\mathcal{A}}$ for three α-particles is very similar to the antisymmetrization operator $\hat{A}$ of three-dineutron system:

[2] We denoted an antisymmetrization operator of 3α system by calligraphic $\hat{\mathcal{A}}$ in order to distinguish it from antisymmetrization operator $\hat{A}$ of $3d'$ system

[3] Such interchange corresponds to the permutation of two boson clusters.

$$\hat{\mathcal{A}}_1 = 6\hat{\mathcal{A}}_{11} + 6\hat{\mathcal{A}}_{12} + 3\left(\hat{\mathcal{A}}^0_{13} + \hat{\mathcal{A}}^+_{13} + \hat{\mathcal{A}}^-_{13}\right),$$

$$\hat{\mathcal{A}}_{11}f(\mathbf{A},\mathbf{B}) = f(\mathbf{A},\mathbf{B}),\quad \hat{\mathcal{A}}_{12}f(\mathbf{A},\mathbf{B}) = 2f\left(-\frac{\mathbf{A}}{2},-\frac{\mathbf{B}}{2}\right),\quad \hat{\mathcal{A}}^0_{13}f(\mathbf{A},\mathbf{B}) = 2f\left(\frac{\mathbf{A}+\mathbf{B}}{2},\frac{\mathbf{A}+\mathbf{B}}{2}\right).$$

Operators $\hat{A}^{\pm}_{ij}$ are defined as follows:

$$\hat{\mathcal{A}}^{\pm}_{ij}f(\mathbf{A},\mathbf{B}) = \left[\hat{A}^0_{ij}f(\mathbf{A},\mathbf{B})\right]_{\mathbf{A}\to\exp\left(\pm i\frac{2\pi}{3}\right)\mathbf{A},\mathbf{B}\to\exp\left(\mp i\frac{2\pi}{3}\right)\mathbf{B}}.$$

Permutations belonging to this group differ from the permutations in three-dineutron system only in one respect: the latter refer to fermions, while the former relate to bosons. Considering dineutrons and diprotons as bosons, we come to the conclusion that $\hat{A}_{11}$ corresponds to the permutations of α-clusters, $\hat{A}_{12}$ can be matched with the exchange of dineutrons and diprotons between all α-particles, while $\hat{A}^0_{13}$ and $\hat{A}^{\pm}_{13}$ are associated with the permutations of dineutrons and diprotons between two α-clusters with the third cluster being a spectator.

The second part $\hat{\mathcal{A}}_2$ of the antisymmetrization operator of 3α system

$$\hat{\mathcal{A}}_2 = 6\hat{\mathcal{A}}_{21} + 6\hat{\mathcal{A}}_{22} + 3(\hat{\mathcal{A}}^0_{23} + \hat{\mathcal{A}}^+_{23} + \hat{\mathcal{A}}^-_{23})$$

combines permutations involving deuteron clusters:

$$\hat{\mathcal{A}}_{21}f(\mathbf{A},\mathbf{B}) = 4f\left(-\frac{\mathbf{A}}{2},-\frac{\mathbf{B}}{2}\right),\ \hat{\mathcal{A}}_{22}f(\mathbf{A},\mathbf{B}) = 8f\left(\frac{\mathbf{A}}{4},\frac{\mathbf{B}}{4}\right),\quad \hat{\mathcal{A}}^0_{23}f(\mathbf{A},\mathbf{B}) = 8f\left(-\frac{\mathbf{A}+\mathbf{B}}{4},-\frac{\mathbf{A}+\mathbf{B}}{4}\right);$$

$\hat{\mathcal{A}}_{21}$ includes permutations of deuteron clusters with unit spin projection; $\hat{\mathcal{A}}_{22}$ is associated with the exchange of nucleons between three deuterons, while the remaining three deuterons (characterized by zero spin projection) are spectators; $\hat{\mathcal{A}}^0_{23}, \hat{\mathcal{A}}^{\pm}_{23}$ combine permutations resulting in the destruction of one α-cluster with the subsequent distribution of two constituent deuterons with unit spin projection between two other α-clusters and formation of the cluster from two remaining deuterons with zero spin projections.

Operator $\hat{\mathcal{A}}_3$

$$\hat{\mathcal{A}}_3 = 6\left(\hat{\mathcal{A}}_{31} + \hat{\mathcal{A}}_{32} + \hat{\mathcal{A}}_{33} + \hat{\mathcal{A}}^0_{34} + \hat{\mathcal{A}}^+_{34} + \hat{\mathcal{A}}^-_{34}\right)$$

includes permutations of protons, with dineutrons remaining intact:

$$\hat{\mathcal{A}}_{31}f(\mathbf{A},\mathbf{B}) = 4f\left(\frac{\mathbf{A}}{4},\frac{\mathbf{B}}{4}\right);\quad \hat{\mathcal{A}}_{32}f(\mathbf{A},\mathbf{B}) = 4f\left(\frac{5+i\sqrt{3}}{8}\mathbf{A},\frac{5-i\sqrt{3}}{8}\mathbf{B}\right);$$

$$\hat{\mathcal{A}}_{33}f(\mathbf{A},\mathbf{B}) = 4f\left(\frac{5-i\sqrt{3}}{8}\mathbf{A},\frac{5+i\sqrt{3}}{8}\mathbf{B}\right);\quad \hat{\mathcal{A}}^0_{34}f(\mathbf{A},\mathbf{B}) = 4f\left(\frac{2\mathbf{A}-\mathbf{B}}{4},-\frac{\mathbf{A}-2\mathbf{B}}{4}\right).$$

$\hat{\mathcal{A}}_{31}$ combines simultaneous cyclic permutations of spin-up and spin-down protons; $\hat{\mathcal{A}}_{32}$ and $\hat{\mathcal{A}}_{33}$ contain cyclic permutations of spin-down or spin-up protons, accordingly; $\hat{\mathcal{A}}^0_{34}, \hat{\mathcal{A}}^{\pm}_{34}$ are responsible for the one-proton exchange between one cluster and each of the remaining two clusters.

The next part of antisymmetrization operator $\hat{\mathcal{A}}$ is related to the exchange of spin-down protons between two clusters accompanied by the identity or cyclic permutations of dineutrons.

$$\hat{\mathcal{A}}_4 = 6(\hat{\mathcal{A}}^0_{41} + \hat{\mathcal{A}}^+_{41} + \hat{\mathcal{A}}^-_{41} + \hat{\mathcal{A}}^0_{42} + \hat{\mathcal{A}}^+_{42} + \hat{\mathcal{A}}^-_{42} + \hat{\mathcal{A}}^0_{43} + \hat{\mathcal{A}}^+_{43} + \hat{\mathcal{A}}^-_{41})$$

In this case dineutrons and one of diprotons are not rearranged.

$$\hat{\mathcal{A}}^0_{41}f(\mathbf{A},\mathbf{B}) = -4f\left(\frac{3\mathbf{A}+\mathbf{B}}{4},\frac{\mathbf{A}+3\mathbf{B}}{4}\right),$$

$$\hat{\mathcal{A}}^0_{42} f(\mathbf{A},\mathbf{B}) = -4f\left(\frac{i\sqrt{3}\mathbf{A}+\mathbf{B}}{4}, \frac{\mathbf{A}-i\sqrt{3}\mathbf{B}}{4}\right), \quad \hat{\mathcal{A}}^0_{43} f(\mathbf{A},\mathbf{B}) = -4f\left(\frac{-i\sqrt{3}\mathbf{A}+\mathbf{B}}{4}, \frac{\mathbf{A}+i\sqrt{3}\mathbf{B}}{4}\right).$$

$\hat{\mathcal{A}}_{41}$, $\hat{\mathcal{A}}_{42}$ and $\hat{\mathcal{A}}_{43}$ are associated with the identity, clockwise or contraclockwise cyclic permutations of dineutrons, accordingly.

Operator $\hat{\mathcal{A}}_5$ is characterized by the odd number of pair permutations of nucleons:

$$\hat{\mathcal{A}}_5 = 6(3\hat{\mathcal{A}}_{51} + \hat{\mathcal{A}}^0_{52} + \hat{\mathcal{A}}^+_{52} + \hat{\mathcal{A}}^-_{52} + \hat{\mathcal{A}}^0_{53} + \hat{\mathcal{A}}^+_{53} + \hat{\mathcal{A}}^-_{53}).$$

In this case one diproton remains a spectator.

$$\hat{\mathcal{A}}_{51} f(\mathbf{A},\mathbf{B}) = -8f\left(\frac{\mathbf{A}}{4}, \frac{\mathbf{B}}{4}\right), \quad \hat{\mathcal{A}}^0_{52} f(\mathbf{A},\mathbf{B}) = -8f\left(\frac{1}{4}\mathbf{A} + \frac{i\sqrt{3}}{4}e^{-i2\pi/3}\mathbf{B}, -\frac{i\sqrt{3}}{4}e^{i2\pi/3}\mathbf{A} + \frac{1}{4}\mathbf{B}\right),$$

$$\hat{\mathcal{A}}^0_{53} f(\mathbf{A},\mathbf{B}) = -8f\left(\frac{1}{4}\mathbf{A} - \frac{i\sqrt{3}}{4}e^{i2\pi/3}\mathbf{B}, \frac{i\sqrt{3}}{4}e^{-i2\pi/3}\mathbf{A} + \frac{1}{4}\mathbf{B}\right).$$

$\hat{\mathcal{A}}_{51}$ combines permutations resulting in the two-nucleon exchange between all clusters; $\hat{\mathcal{A}}^0_{52}, \hat{\mathcal{A}}^\pm_{52}$ and $\hat{\mathcal{A}}^0_{53}, \hat{\mathcal{A}}^\pm_{53}$ correspond to the one-nucleon exchange between two clusters accompanied by the exchange of deuterons with unit or zero spin projection, accordingly.

The last part $\hat{\mathcal{A}}_6$ of the antisymmetrization operator looks like

$$\hat{\mathcal{A}}_6 = 3(\hat{\mathcal{A}}^0_{61} + \hat{\mathcal{A}}^+_{61} + \hat{\mathcal{A}}^-_{61} + \hat{\mathcal{A}}^0_{62} + \hat{\mathcal{A}}^+_{62} + \hat{\mathcal{A}}^-_{62}) + 6(\hat{\mathcal{A}}^0_{63} + \hat{\mathcal{A}}^+_{63} + \hat{\mathcal{A}}^-_{63}).$$

All constituent parts of operator $\hat{\mathcal{A}}_6$ can be expressed via previously defined permutation operators:

$$\hat{\mathcal{A}}^0_{61} = 2\hat{\mathcal{A}}^0_{13}, \quad \hat{\mathcal{A}}^0_{62} = 2\hat{\mathcal{A}}^0_{23}, \quad \hat{\mathcal{A}}^0_{63} = 2\hat{\mathcal{A}}^0_{34}.$$

Taking into account that $\hat{\mathcal{A}}_{21} = 2\hat{\mathcal{A}}_{12}$ and $\hat{\mathcal{A}}_{22} = 2\hat{\mathcal{A}}_{31}$, we come to the final expression for the antisymmetrization operator of the 3α system:

$$\begin{aligned}\hat{\mathcal{A}} = &\sum_{i=1}^{6}\frac{1}{3!}\hat{\mathcal{A}}_i = \hat{\mathcal{A}}_{11} + 3\hat{\mathcal{A}}_{12} + 3\hat{\mathcal{A}}_{31} + \hat{\mathcal{A}}_{32} + \hat{\mathcal{A}}_{33} + 3\hat{\mathcal{A}}_{51} + \\ &+ \frac{3}{2}(\hat{\mathcal{A}}^0_{13} + \hat{\mathcal{A}}^+_{13} + \hat{\mathcal{A}}^-_{13}) + \frac{3}{2}(\hat{\mathcal{A}}^0_{23} + \hat{\mathcal{A}}^+_{23} + \hat{\mathcal{A}}^-_{23}) + 3(\hat{\mathcal{A}}^0_{34} + \hat{\mathcal{A}}^+_{34} + \hat{\mathcal{A}}^-_{34}) + \\ &+ \hat{\mathcal{A}}^0_{41} + \hat{\mathcal{A}}^+_{41} + \hat{\mathcal{A}}^-_{41} + \hat{\mathcal{A}}^0_{42} + \hat{\mathcal{A}}^+_{42} + \hat{\mathcal{A}}^-_{42} + \hat{\mathcal{A}}^0_{43} + \hat{\mathcal{A}}^+_{43} + \hat{\mathcal{A}}^-_{43} + \hat{\mathcal{A}}^0_{52} + \hat{\mathcal{A}}^+_{52} + \hat{\mathcal{A}}^-_{52} + \hat{\mathcal{A}}^0_{53} + \hat{\mathcal{A}}^+_{53} + \hat{\mathcal{A}}^-_{53}.\end{aligned}$$

Applying operator $\hat{\mathcal{A}}$ to the basis functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}(\mathbf{A},\mathbf{B})$, we shall come to the following expansion of the partial norm kernel $i^{3\alpha}_{(2\nu-4\mu,2\mu)}$ for the 3α-system:

$$\begin{aligned}i^{3\alpha}_{(2\nu-4\mu,2\mu)} = &\sum_{q=0}^{[\frac{\nu-2\mu}{3}]}\sum_{\tilde{q}=0}^{[\frac{\nu-2\mu}{3}]-q}\langle 2\nu,\nu-3q-2\mu|2\nu,\nu-3q-3\tilde{q}-2\mu\rangle_{3\alpha}\times \\ &\times\left(\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\tilde{\psi}^{\nu-3q-3\tilde{q}-2\mu}_{(2\nu-4\mu,2\mu)} + \psi^{\nu-3q-3\tilde{q}-2\mu}_{(2\nu-4\mu,2\mu)}\tilde{\psi}^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}\right). \qquad (12)\end{aligned}$$

The above expansion differs from similar expression (8) for the three-dineutron system only in the explicit form of the matrix elements of the secular equation:

$$\begin{aligned}&\langle 2\nu,\nu-3q-2\mu|2\nu,\nu-3q-3\tilde{q}-2\mu\rangle_{3\alpha} = \\ &= \left(1+\frac{6}{4^\nu}\left(1-\frac{2}{4^\nu}\right)+\frac{1}{8^{6q-1}}\left(\frac{7}{16}\right)^{\nu-3q}\sum_{m=0}^{3q}\binom{6q}{2m}(-1)^m 5^{6q-2m}3^m\right)\delta_{\tilde{q},0} +\end{aligned}$$

$$+\frac{9}{4^{2\nu-1}}\sqrt{\frac{(\nu+3q+3\tilde{q}-2\mu)!(\nu-3q-3\tilde{q}-2\mu)!}{(\nu+3q-2\mu)!(\nu-3q-2\mu)!(1+\delta_{q,0})(1+\delta_{\tilde{q},-q})}}\begin{pmatrix}\nu+3q-2\mu\\ \nu-3q-3\tilde{q}-2\mu\end{pmatrix}\times$$

$$\times\left\{3^{2\mu}(-1)^{\tilde{q}}\left[2^{6q+3\tilde{q}}{}_2F_1(\alpha,\beta;\gamma;4)+2^{2\nu-6q-3\tilde{q}-4\mu}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{4}\right)\right]-\right.$$

$$-\frac{1}{3}8^{2\mu}\left[3^{6q+3\tilde{q}}{}_2F_1(\alpha,\beta;\gamma;9)+3^{2\nu-6q-3\tilde{q}-4\mu}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{9}\right)\right]-$$

$$\left.-2^{2\mu}\left(1+(-1)^{\tilde{q}}\right)\left[(i\sqrt{3})^{6q+3\tilde{q}}{}_2F_1(\alpha,\beta;\gamma;3)+3^{\nu-2\mu}(-1)^{q}(i\sqrt{3})^{-6q-3\tilde{q}}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{3}\right)\right]\right\}\Bigg),$$

where indices of hypergeometric function are defined as follows:

$$\alpha=-\nu+3q+2\mu,\quad \beta=-\nu+3q+3\tilde{q}+2\mu,\quad \gamma=6q+3\tilde{q}+1.$$

Eigenfunctions $\Psi_{(2\nu-4\mu,2\mu)}$ of the antisymmetrization operator $\hat{\mathcal{A}}$ for the 3α-system are given in terms of orthogonal polynomials of a discrete variable, as in the case of the $3d'$-system (see Eq. (7)). The number of different states possessing SU(3) symmetry $(2\nu-4\mu,2\mu)$, i.e., the degree of the SU(3)-degeneracy, is the same for three-boson system, $3d'$-system and 3α-system and equal to $[\frac{\nu-2\mu}{3}]+1$. However, some of these eigenfunctions correspond to zero eigenvalues and thus they are forbidden by the Pauli principle. The number of the latter functions depends on the system considered and increases with increasing the number of nucleons in each cluster. For example, in the three-boson system all the states belonging to the $(2\nu-4\mu,2\mu)$ representation have been allowed by the Pauli principle, while in the three-dineutron system at a given number of oscillator quanta ν there is a Pauli-forbidden state $\Psi_{(2\nu,0)_{k=0}}$. In the 3α-system at $\nu\geq 3$ there are two Pauli-forbidden states belonging to the $(2\nu,0)$ SU(3)-representation and one Pauli-forbidden state characterized by the $(2\nu-4,2)$ SU(3)-indices. All the Pauli-allowed states of the 3α-system have distinct

Table 1
Eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$ belonging to the first three families of the Pauli-allowed states of the 3α-system

	$2k=4$	$2k=6$	$2k=8$
ν	$\mu=2$ $\mu=1$ $\mu=0$	$\mu=3$ $\mu=2$ $\mu=1$ $\mu=0$	$\mu=4$ $\mu=3$ $\mu=2$ $\mu=1$ $\mu=0$
4	0.5933		
5	0.6748 0.4894		
6	0.7161 0.5647 0.6497	0.8714	
7	0.7366 0.6333 0.6269	0.9081 0.8657	
8	0.7450 0.6859 0.6457	0.9246 0.8852 0.9073	0.9640
9	0.7482 0.7196 0.6773	0.9324 0.9042 0.8959 0.9478	0.9754 0.9666
10	0.7493 0.7372 0.7071	0.9356 0.9191 0.9000 0.9193	0.9805 0.9712 0.9770
11	0.7497 0.7449 0.7283	0.9368 0.9287 0.9106 0.9084	0.9828 0.9759 0.9738 0.9865
12	0.7499 0.7480 0.7404	0.9372 0.9338 0.9212 0.9093	0.9838 0.9796 0.9747 0.9792 0.9906
13	0.7500 0.7492 0.7461	0.9374 0.9360 0.9290 0.9159	0.9842 0.9821 0.9774 0.9763 0.9837
14	0.7500 0.7497 0.7485	0.9375 0.9369 0.9336 0.9236	0.9843 0.9834 0.9801 0.9764 0.9792
15	0.7500 0.7499 0.7494	0.9375 0.9373 0.9358 0.9298	0.9843 0.9840 0.9821 0.9783 0.9773
16	0.7500 0.7499 0.7498	0.9375 0.9374 0.9368 0.9337	0.9844 0.9842 0.9833 0.9804 0.9775
17	0.7500 0.7500 0.7499	0.9375 0.9375 0.9372 0.9358	0.9844 0.9843 0.9839 0.9822 0.9790
18	0.7500 0.7500 0.7500	0.9375 0.9375 0.9374 0.9368	0.9844 0.9844 0.9842 0.9833 0.9808
19	0.7500 0.7500 0.7500	0.9375 0.9375 0.9375 0.9372	0.9844 0.9844 0.9843 0.9839 0.9823
20	0.7500 0.7500 0.7500	0.9375 0.9375 0.9375 0.9374	0.9844 0.9844 0.9843 0.9842 0.9833

eigenvalues, which tend to the eigenvalues of a 2α subsystem with increasing the number of oscillator quanta ν (see Table 1):

$$\lim_{\nu-2\mu\to\infty}\Lambda^{^{12}\mathrm{C}=3\alpha}_{(2\nu-4\mu,2\mu)_k}\to\lambda^{^{8}\mathrm{Be}=2\alpha}_{2k}=1-\left(\frac{1}{4}\right)^{k-1}+3\delta_{k,0}.$$

Eigenfunctions $\Psi_{(2\nu-4\mu,2\mu)_k}$ also become more complicated with increasing the number of nucleons. In the case of 3α-system each eigenfunction of the antisymmetrization operator is a superposition of functions $\psi^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)}$ with all nonvanishing expansion coefficients $D^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)_k}$. However, as the eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$ of the 3α norm kernel approach limit values $\lambda^{^{8}\mathrm{Be}}_{2k}$, corresponding eigenvectors $\Psi_{(2\nu-4\mu,2\mu)_k}$ take simple analytical form $\Psi^{\mathrm{as}}_{(2\nu-4\mu,2\mu)_k}$ (see Eq. (9)), as can be seen from Table 2. Consequently, in the limit $\nu\gg k$ the expansion coefficients $D^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)_k}$ can be identified with the Kravchuk polynomials of a discrete variable:

$$D^{\nu-3q-2\mu}_{(2\nu-4\mu,2\mu)_k}\to\sqrt{\frac{6}{1+\delta_{q,0}}}\,\bar{\mathcal{K}}^{1/2}_{2k}(\nu-3q-2\mu),$$

where the explicit form of the Kravchuk polynomials $\bar{\mathcal{K}}^{1/2}_{2k}(\nu-3q-2\mu)$ is given by Eq. (10).

Therefore, index k, which makes sense of the number of oscillator quanta accounted for by the ^{8}Be subsystem, serves as an additional quantum number for the SU(3)-degenerated states. It should be noted also that due to the incoincidence of eigenvalues $\Lambda^{^{12}\mathrm{C}=3\alpha}_{(2\nu-4\mu,2\mu)_k}$, the eigenfunctions $\Psi_{(2\nu-4\mu,2\mu)_k}$ are uniquely determined. Any unitary transformation applied to the latter SU(3)-basis functions would disrupt the diagonal form of the norm kernel of the 3α system and, hence, is inappropriate in this case.

Noteworthy also is a remarkably simple form of the eigenfunctions $\Psi_{(2\nu-4\mu,2\mu)_{k=\mu}}$ in the case $\nu-2\mu<3$:

$$\Psi_{(2\nu-4\mu,2\mu)_{k=\mu}}=\psi^{\nu-2\mu}_{(2\nu-4\mu,2\mu)},\quad\left[\frac{\nu-2\mu}{3}\right]=0.$$

It is precisely this fact that explains why the overlap integrals $\int\Psi_{(2\nu-4\mu,2\mu)_{k=\mu}}\Psi^{\mathrm{as}}_{(2\nu-4\mu,2\mu)_{k=\mu}}d\mu_B$ are equal to unity, as long as $\nu-2\mu<3$.

In turn, the eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_{k=\mu}}$ are given by the following analytical expression, provided that $\nu-2\mu<3$:

$$\begin{aligned}\Lambda_{(2\nu-4\mu,2\mu)_{k=\mu}}=&\,1+\frac{6}{4^\nu}\left(1-\frac{2}{4^\nu}\right)+8\left(\frac{7}{16}\right)^\nu+\frac{18}{4^{2\nu}}\times\\&\times\left\{3^{2\mu}\left[{}_2F_1(-\nu+2\mu,-\nu+2\mu;1;4)+4^{\nu-2\mu}{}_2F_1\left(-\nu+2\mu,-\nu+2\mu;1;\frac{1}{4}\right)\right]-\right.\\&-\frac{1}{3}8^{2\mu}\left[{}_2F_1(-\nu+2\mu,-\nu+2\mu;1;9)+9^{\nu-2\mu}{}_2F_1\left(-\nu+2\mu,-\nu+2\mu;1;\frac{1}{9}\right)\right]-\\&\left.-\,2^{2\mu+1}\left[{}_2F_1(-\nu+2\mu,-\nu+2\mu;1;3)+3^{\nu-2\mu}{}_2F_1\left(-\nu+2\mu,-\nu+2\mu;1;\frac{1}{3}\right)\right]\right\}\end{aligned}$$

From Table 1 and Table 2 we notice that the Pauli-allowed basis states $\Psi_{(2\nu-4\mu,2\mu)_k}$ can be arranged into the branches and families, with all the states of a particular branch having common SU(3)-symmetry index μ and overlapping generously with corresponding asymptotic function (9), but differing in value of the first index $\bar{\lambda}$ of the SU(3)-symmetry. The eigenvalues belonging to a given branch tend to the same limit value $\lambda^{^{8}\mathrm{Be}}_{2k}$ with the number of quanta increasing. The branches which share limit eigenvalues are combined in the family of the eigenstates, which thus is completely determined by the degree k of the corresponding Kravchuk polynomial.

It should be noted that the states listed in Table 1 and Table 2 exhaust all possible basis functions allowed by the Pauli principle, as long as $\nu\leq 9$. As ν increases, new families of states characterized by $2k>8$ also

Table 2
Overlap integrals $\int \Psi_{(2\nu-4\mu,2\mu)_k} \Psi^{\rm as}_{(2\nu-4\mu,2\mu)_k} d\mu_B$ versus the number of quanta ν for the first three families of the Pauli-allowed states of the 3α-system

	$2k=4$	$2k=6$	$2k=8$
ν	$\mu=2$ $\mu=1$ $\mu=0$	$\mu=3$ $\mu=2$ $\mu=1$ $\mu=0$	$\mu=4$ $\mu=3$ $\mu=2$ $\mu=1$ $\mu=0$
4	1.0000		
5	1.0000 0.9670		
6	1.0000 0.9957 0.8688	1.0000	
7	0.9988 0.9994 0.9661	1.0000 0.9535	
8	0.9996 0.9944 0.9919	1.0000 0.9929 0.8127	1.0000
9	0.9999 0.9940 0.9960	0.9977 0.9990 0.9321 0.7255	1.0000 0.9474
10	1.0000 0.9973 0.9949	0.9993 0.9932 0.9800 0.8677	1.0000 0.9914 0.8064
11	1.0000 0.9993 0.9958	0.9998 0.9923 0.9920 0.9421	0.9969 0.9987 0.9269 0.7151
12	1.0000 0.9998 0.9981	1.0000 0.9965 0.9912 0.9790	0.9989 0.9928 0.9778 0.8537 0.6447
13	1.0000 1.0000 0.9994	1.0000 0.9990 0.9924 0.9908	0.9998 0.9920 0.9913 0.9311 0.8009
14	1.0000 1.0000 0.9998	1.0000 0.9998 0.9962 0.9925	1.0000 0.9963 0.9907 0.9736 0.8826
15	1.0000 1.0000 1.0000	1.0000 1.0000 0.9988 0.9942	1.0000 0.9990 0.9919 0.9885 0.9399
16	1.0000 1.0000 1.0000	1.0000 1.0000 0.9997 0.9970	1.0000 0.9998 0.9959 0.9909 0.9741
17	1.0000 1.0000 1.0000	1.0000 1.0000 0.9999 0.9988	1.0000 0.9999 0.9987 0.9928 0.9878
18	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000 0.9997	1.0000 1.0000 0.9996 0.9962 0.9914
19	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000 0.9999	1.0000 1.0000 0.9999 0.9986 0.9937
20	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000 1.0000	1.0000 1.0000 1.0000 0.9996 0.9966

appear. However, eigenstates and eigenvalues belonging to such families are governed by the same laws as those tabulated in Tables 1, 2. The state $\Psi_{(0,4)_{k=2}}$ characterized by the minimum number of quanta $\nu=4$ and belonging to the $\mu=2$ branch of the $k=k_{\rm min}=2$ family of the Pauli-allowed states is identical to the state of the leading SU(3)-representation of the oscillator translation-invariant shell model in the Elliott scheme. Other states belonging to the $\mu=2$ branch provide the main corrections to the states of the Elliott model resulting from the α-clustering of the ^{12}C nucleus and its decay through the ^{12}C$\rightarrow^{8}$Be+α channel. Branches of the states belonging to the families characterized by $k>k_{\rm min}$ reproduce the excitations of the ^{8}Be accompanied by the increasing of the number of oscillator quanta accounted for by this subsystem.

Existence of different limit eigenvalues $\lambda^{^8{\rm Be}}_{2k}$ reflects the possibility for two α-clusters to be close to each other and far apart from the third α-cluster. Owing to this the unitary transformation from the SU(3)-basis to the other one is rendered possible only within a particular family of the Pauli-allowed states. Transformation to the three-cluster hyperspherical harmonics, contrastingly, involves the states belonging to different families. The hypermomentum can not serve as quantum number of Pauli-allowed basis functions, because the Pauli exclusion principle mixes the states with different values of hypermomentum. Hence, restriction for the maximum value of hypermomentum K included in calculation $K\leq K_{max}$ leads to spoiling the Pauli-allowed basis functions corresponding to the number of oscillator quanta $\nu>K_{max}$, and this effect is enhanced with increasing ν. The angular-momentum-coupled basis can be used instead in the region $\nu\gg k$, where all eigenvalues $\Lambda_{(2\nu-4\mu,2\mu)_k}$ belonging to the $k_{\rm th}$ family coincide with their limit values $\lambda^{^8{\rm Be}}_{2k}$. Basis functions of the latter basis are labeled by the number of quanta, and the angular momenta of the ^{8}Be subsystem coinciding with the angular momentum of the relative motion of this subsystem and a remaining α-particle [4]. The transformation to this basis is performed within a particular family of the Pauli-allowed states and, therefore, does not result in violation of the Pauli principle. At the same time, the angular-momentum-coupled basis allows one to set the asymptotic boundary conditions for the expansion

[4] Recall that only the states with total angular momentum $L=0$ are considered in this paper.

coefficients of a three-cluster wave function in the continuum. This point has been discussed in detail in Ref. [13] with the ^{3}H+$n+n$ system.

From the above discussion it appears that asymptotically each family of Pauli-allowed states of 3α-system corresponds to a certain binary decay channel of the latter system into a 2α subsystem occurring in a ground or an excited harmonic-oscillator state and a remaining α-particle. Hence, the excited states of the ^{12}C nucleus should decay via subsequent stage ^{12}C$\rightarrow^8$Be+$\alpha \rightarrow \alpha+\alpha+\alpha$ rather than in a "democratic" way ^{12}C$\rightarrow \alpha+\alpha+\alpha$ usually associated with hyperspherical harmonics. The latter basis can, probably, provide an adequate description of the ground state of the ^{12}C nucleus, but it is definitely not appropriate for studying the continuum states.

Finally, let us compare the behavior of the eigenvalues and eigenfunctions of the 3α system with those of the ^{5}H=^{3}H+$n+n$ system discussed in detail in Ref. [13]. First of all, at a given number of quanta ν the number of the Pauli-allowed states in the 3α system is much less than that of the ^{5}H system. In the 3α system only the families of the states corresponding to even values of quantum number k are consistent with the requirements of the Pauli exclusion principle, while the model space of the Pauli-allowed states of ^{3}H+$n+n$ system includes the families characterized by both odd and even values of k. The number of the Pauli-forbidden states in the 3α-system is also greater than in the ^{5}H nucleus. In the ^{5}H system only one branch falls into the category of the Pauli-forbidden states. It is characterized by $k=0$ and corresponds to the occupation of an s-state in the ^{4}H subsystem by one of the valence neutrons. The 3α system have three branches of the Pauli-allowed states: a branch being a member of $k=0$ family and two branches entering in the family $2k=2$. These two families correspond to two forbidden states of the 2α subsystem. Moreover, whereas in the case of the ^{5}H nucleus all branches belonging to the same family appear at the same number of quanta, the branches combining into a particular family of the 3α system start in one-quantum increment. A new family of the Pauli-allowed states in the ^{5}H system arises with increasing the number of quanta by one, while families of such states in the 3α system emerge in three-quantum interval. This results in decreasing the model space of the Pauli-allowed functions in the internal region of small distance between clusters in the 3α system as compared with the ^{5}H system.

All the differences discussed above are due to a high symmetry of the 3α-system, which comprises of three identical clusters, while the ^{3}H+$n+n$ system have only two identical constituents. In addition, in the 3α-system each family begins with the branch having maximum possible SU(3)-symmetry index $\mu=k$. This fact gives grounds to expect that the branches characterized by $\mu=k$ should be more favorable than the branches with $\mu=0$, in contrast to the case of the ^{5}H nucleus. Of course, both for the case of the 3α system and the ^{3}H+$n+n$ system the families of the allowed states associated with the low-order Kravchuk polynomials should dominate.

The eigenvalues of the antisymmetrization operator of the 3α system tend to unity from below resulting in the repulsive effective interaction between α-clusters due to the change in the kinetic energy of relative motion under the effect of the Pauli exclusion principle (see Refs. [13,19] for details). Hence taking into account all the exchange effects in 3α-system may allows one to overcome the problem of overbinding of the ^{12}C nucleus without introducing an additional three-cluster repulsion.

7. Conclusion

Within the microscopic model based on the algebraic version of the resonating group method the role of the Pauli principle in the formation of continuum wave function of nuclear systems composed of three identical s-clusters has been investigated. Our principal concern has been with the study of the exchange effects contained in the genuine three-cluster norm kernel, i.e., taking into account the eigenvalues of the Pauli-allowed states.

Norm kernels for the system of three identical fermions and the system of three identical bosons have been constructed in the Fock-Bargmann space. It has been shown that the former serves as the main building block for the norm kernels of nuclear systems composed of three identical clusters, while the latter generates the basis functions needed for the description of nuclear systems composed of three identical boson clusters (such as $^6n=^2n+^2n+^2n$ and ^{12}C=$\alpha+\alpha+\alpha$). Simple analytical method of constructing the norm kernel

for $3d'$ system and 3α system in the Fock-Bargmann space has been suggested and realized.

The Fock-Bargmann image of the antisymmetrization operator has been found for three-fermion, three-boson, three-dineutron and 3α systems. Classification of different constituents of the antisymmetrization operator by belonging to a certain group of permutations is given. Careful analysis of the structure of the eigenfunctions and behavior of the eigenvalues of the antisymmetrizer has been performed for all the above-mentioned three-cluster systems. It has been demonstrated that the Pauli-allowed states of three-cluster systems composed of identical boson clusters can be represented as a superposition of the eigenfunctions of the three-boson norm kernel, with the expansion coefficients being the orthogonal polynomials of a discrete variable.

The Pauli-allowed basis functions for the 3α and the $3d'$ systems are given in an explicit form and the asymptotic behavior of these functions is established. Eigenvalues of the three-cluster system are shown to tend to the eigenvalues of a two-cluster subsystem with increasing the number of oscillator quanta ν. At the same time, the corresponding eigenvectors take simple analytical form, while the expansion coefficients become the Kravchuk polynomials as the number of oscillator quanta increases. We suggest a way of resolving the problem of the SU(3) degeneracy of the Pauli-allowed states. A degree of the Kravchuk polynomial serves as an additional quantum number to label the states belonging to the same SU(3) irreducible representations.

Complete classification of the eigenfunctions and the eigenvalues of the ^{12}C norm kernel by the ^{8}Be$=\alpha+\alpha$ eigenvalues has been given for the first time. We have demonstrated that for the 3α system such classification is unique in that it is consistent with the requirements of the Pauli exclusion principle both in the region of small intercluster distances and in the asymptotic region where the scattering matrix elements are produced. Due to incoincidence of eigenvalues of the antisymmetrization operator of the 3α system the corresponding eigenfunctions are uniquely determined. Any unitary transformation applied to the latter SU(3)-basis functions would disrupt the diagonal form of the norm kernel of 3α system and, hence, is inappropriate in this case. Contrastingly, the eigenfunctions of the three-neutron or the three-boson systems are determined only to the unitary transformation, because all the Pauli-allowed states have unit eigenvalues. Hence, in this case the SU(3)-classification is only one of possible variants. The $3d'$ system is a borderline case, because the eigenvalues of the norm kernel are equal and tend to unity for all the Pauli-allowed states. Consequently, any transformation, which involves only the Pauli-allowed states and remains the Pauli-forbidden states intact, does not disrupt the diagonal form of the norm kernel.

Summarizing, the Pauli-allowed states of the 3α system can be arranged into the branches and the families. The eigenvalues belonging to a given branch tend to the same limit value $\lambda_{2k}^{^{8}\mathrm{Be}}$ of the 2α subsystem with the number of quanta increasing. The branches which share limit eigenvalues are combined in the family of the eigenstates, which asymptotically corresponds to a certain binary decay channel of the 3α system into a 2α subsystem occurring in a particular harmonic-oscillator state and a remaining α-particle. Hence, the excited states of the ^{12}C nucleus should decay via the subsequent stage $^{12}\mathrm{C}\rightarrow{}^{8}\mathrm{Be}+\alpha\rightarrow\alpha+\alpha+\alpha$ rather than in a "democratic" way $^{12}\mathrm{C}\rightarrow\alpha+\alpha+\alpha$ usually associated with hyperspherical harmonics.

Proper truncation of the model space of the Pauli-allowed basis functions consists in neglecting families of the states with high values of additional quantum number k and using the angular-momentum-coupled basis in the region $\nu\gg k$ for setting the boudary conditions for the three-cluster wave function in the continuum.

Spectrum of the ^{12}C norm kernel has also been compared to that of the ^{5}H system. The model space of the Pauli-allowed functions in the internal region of small distance between clusters in 3α system has been shown to decrease essentially as compared with the ^{5}H system due to a high symmetry of 3α-system, which comprises of three identical clusters.

Appendix A. Partial norm kernels of the 3α system

The norm kernel of the 3α system can be divided into six partial norm kernels, each corresponding to a certain group of nucleon permutations:

$$I_{3\alpha}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right)=\sum_{i=1}^{6}\frac{1}{3!}\hat{\mathcal{A}}_i\exp\left\{(\mathbf{A}\tilde{\mathbf{A}})+(\mathbf{B}\tilde{\mathbf{B}})\right\}=\sum_{i=1}^{6}\frac{1}{3!}I_{3\alpha}^{(i)}\left(\mathbf{A},\mathbf{B};\tilde{\mathbf{A}},\tilde{\mathbf{B}}\right),$$

where

$$I_{3\alpha}^{(1)} = \frac{1}{3!}(x_1+y_1+z_1+x_2+y_2+z_2)^2, \quad I_{3\alpha}^{(2)} = \frac{1}{3!}(\bar{x}_1+\bar{y}_1+\bar{z}_1+\bar{x}_2+\bar{y}_2+\bar{z}_2)^2$$
$$I_{3\alpha}^{(3)} = \frac{1}{3!}2(x_1+y_1+z_1+x_2+y_2+z_2)(\bar{x}_1+\bar{y}_1+\bar{z}_1+\bar{x}_2+\bar{y}_2+\bar{z}_2)$$
$$I_{3\alpha}^{(4)} = -\frac{1}{3!}4(x_1+y_1+z_1+x_2+y_2+z_2)(x_3+y_3+z_3)(x_4+y_4+z_4)$$
$$I_{3\alpha}^{(5)} = -\frac{1}{3!}4(\bar{x}_1+\bar{y}_1+\bar{z}_1+\bar{x}_2+\bar{y}_2+\bar{z}_2)(x_3+y_3+z_3)(x_4+y_4+z_4)$$
$$I_{3\alpha}^{(6)} = \frac{1}{3!}4(x_3+y_3+z_3)^2(x_4+y_4+z_4)^2.$$

We used the following notations

$$x_1 = \exp\left\{\frac{(\mathbf{A}\tilde{\mathbf{A}})}{2}+\frac{(\mathbf{B}\tilde{\mathbf{B}})}{2}\right\}; \quad x_2 = \exp\left\{\frac{(\mathbf{B}\tilde{\mathbf{A}})}{2}+\frac{(\mathbf{A}\tilde{\mathbf{B}})}{2}\right\};$$
$$y_1 = \exp\left\{e^{i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{A}})}{2}+e^{-i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{B}})}{2}\right\}, \quad y_2 = \exp\left\{e^{i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{A}})}{2}+e^{-i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{B}})}{2}\right\}$$
$$z_1 = \exp\left\{e^{-i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{A}})}{2}+e^{i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{B}})}{2}\right\} \quad z_2 = \exp\left\{e^{-i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{A}})}{2}+e^{i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{B}})}{2}\right\}$$
$$\bar{x}_1 = 2\exp\left\{-\frac{(\mathbf{A}\tilde{\mathbf{A}})}{4}-\frac{(\mathbf{B}\tilde{\mathbf{B}})}{4}\right\}; \quad \bar{x}_2 = 2\exp\left\{-\frac{(\mathbf{B}\tilde{\mathbf{A}})}{4}-\frac{(\mathbf{A}\tilde{\mathbf{B}})}{4}\right\}$$
$$\bar{y}_1 = 2\exp\left\{-e^{i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{A}})}{4}-e^{-i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{B}})}{4}\right\}, \quad \bar{y}_2 = 2\exp\left\{-e^{i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{A}})}{4}-e^{-i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{B}})}{4}\right\}$$
$$\bar{z}_1 = 2\exp\left\{-e^{-i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{A}})}{4}-e^{i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{B}})}{4}\right\}, \quad \bar{z}_2 = 2\exp\left\{-e^{-i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{A}})}{4}-e^{i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{B}})}{4}\right\}$$
$$x_3 = \exp\left\{\frac{(\mathbf{A}\tilde{\mathbf{A}})}{4}+\frac{(\mathbf{B}\tilde{\mathbf{B}})}{4}\right\}; \quad x_4 = \exp\left\{\frac{(\mathbf{B}\tilde{\mathbf{A}})}{4}+\frac{(\mathbf{A}\tilde{\mathbf{B}})}{4}\right\};$$
$$y_3 = \exp\left\{e^{i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{A}})}{4}+e^{-i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{B}})}{4}\right\}, \quad y_4 = \exp\left\{e^{i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{A}})}{4}+e^{-i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{B}})}{4}\right\}$$
$$z_3 = \exp\left\{e^{-i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{A}})}{4}+e^{i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{B}})}{4}\right\} \quad z_4 = \exp\left\{e^{-i\frac{2\pi}{3}}\frac{(\mathbf{B}\tilde{\mathbf{A}})}{4}+e^{i\frac{2\pi}{3}}\frac{(\mathbf{A}\tilde{\mathbf{B}})}{4}\right\}$$

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